DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

INTEGRAL EQUATION AND INTEGRAL TRANSFORM

DEMATH4ELEC4

BLOCK-1

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

INTEGRAL EQUATION AND INTEGRAL TRANSFORM

BLOCK-1

BLOCK-2

BLOCK-1 INTEGRAL EQUATION AND INTEGRAL TRANSFORM

In this Block we will be exploring the concept of Integral Equation.We will discussed the classifications of integral equations such as Fredholm Integral Equation, Volterra Integral Equation and many others along with the kinds of particular equation. We will explore separable kernel and how to solve Fredholm integral equation of the second kind with separable kernel. Also we discussed the method of finding eigenvalue and Eigen function of the fredholm integral equation of the second kind by reducing the equation to an algebraic system of equation. As we know that convergence and uniqueness is an important phenomenon so conditions of convergence and uniqueness of series solution is also discussed. We will enumerate We have studied three different theorems of Fredholm's .

UNIT-1 INTRODUCTION TO INTEGRAL EQUATIONS

STRUCTURE

1.0 OBJECTIVES

Understand the concept of Integral Equation

Understand the Classification of Linear Integral Equations

Comprehend Relations between differential and integral equations

1.1 INTRODUCTION

The subject of Integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ODE(ordinary differential equations) and PDE(partial differential equations) can be transformed into problems of solving some approximate integral equations.

Integral equations were first encountered in the theory of Fourier Integral. In 1826, another integral equation was obtained by Abel. Actual development of the theory of integral equations began with the works of the Italian Mathematician V.Volterra (1896) and the Swedish Mathematician I.Fredholm (1900).

1.2 WHAT IS INTEGRAL EQUATION?

An integral equation is an equation in which the unknown function u(x) to be determined appears under the integral sign. A typical form of an integral equation in u(x) is of the form.

$$g(s) = f(s) + \int_{\alpha(s)}^{\beta(s)} K(s,t)\, g(t)dt \qquad (1.1)$$

Where u(x) is called the kernel of the integral equation α(x) and β(x) are the limit of integration.

For example, for $a \leq s \leq b$; $a \leq t \leq b$, the equations

$$f(s) = \int_a^b K(s,t)\, g(t)dt \qquad (1.2)$$

$$g(s) = f(s) + \int_a^b K(s,t)\, g(t)dt \qquad (1.3)$$

$$g(s) = \int_a^b K(x,t)\, [g(t)]^2 dt \qquad (1.4)$$

In (1), it is easily observed that the unknown function u(x) appears under the integral sign as stated above, and out of the integral sign in most other cases as will be discussed later. It is important to point out that the kernel K (x, t) and the function f(x) in (1) are given in advance. Our goal is to determine u(x) that will satisfy (1), and this may be achieved by using different techniques which will be discussed later.

Integral equation arise naturally in Physics, Chemistry, biology and engineering applications modeled by initial value problems for a finite interval ,. They also arise as representation formulas for the solutions of differential equations. Indeed a differential equation can be replaced by an integral equation that incorporates its boundary condition. As such, each solution of the integral equation automatically satisfies the boundary conditions. Integral equations also form one of the most useful tools in many branches of pure analysis, such as functional analysis and stochastic processes. One can also consider integral equations in which the unknown function is dependent not only on one variable but on several variables. Such for example, is the equation

$$u(s) = f(s) + \int_{\Omega} K(s,t)\, u(t) dt \qquad (1.5)$$

Where , s, t are n - dimensional vectors and Ω is a region of an n −dimensional space. Similarly, one can consider systems of integral equations with several unknown functions. An Integral equation is called **Linear** if only linear operations are performed in it upon the unknown function. Equations (1.2) and (1.3) are linear, while (1.4) is nonlinear. In fact equations (1.2) and (1.3) can be written as

$$L[g(s)] = f(s) \qquad (1.6)$$

Where L[g(s)] = $\int_a^b K(s,t) g(t) dt$. Then for any constant c_1 and c_2, we have i.e. integral equation satisfy linear property.

1.3 CLASSIFICATION OF LINEAR INTEGRAL EQUATIONS

The most frequently used linear integral equations fall under two main classes namely Fredholm and Volterra integral equations. However, in this text we will distinguish four more related types of linear integral equations in addition to the two main classes. In the following is the list of the Fredholm and Volterra integral equations, and the four more related types:

1. Fredholm Integral Equations
2. Volterra Integral Equations
3. Integro-differential Equations
4. Singular Integral Equations
5. Volterra-Fredholm Integral Equations
6. Volterra-Fredholm Integro-differential Equations

1.3.1 Fredholm Linear Integral Equation

The standard form of Fredholm linear integral equations, where the limits of integration **a** and **b** are constants, are given by the form

$$\phi(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt \quad a \leq s,t \leq b \tag{1.8}$$

Where the kernel of the integral equation K(s,t) and the function f(s) are given in advance, and λ is a parameter. The equation (1.8) is called linear because the unknown function g(x) under the integral sign occurs linearly, i.e. the power of g(x) is one. The value of ∅(x) will give the following kinds of Fredholm linear integral equation

1. When ∅(x) = 0, equation (1.8) becomes

$$f(s) + \lambda \int_a^b K(s,t)\, g(t)dt \tag{1.9}$$

And the integral equation is called Fredholm Integral equation of First kind.

2. When ∅(x) = 1, equation (1.8) become

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)dt \qquad (1.10)$$

And the integral equation is called Fredholm integral equation of the second kind. In fact, the equation (1.10) can be obtained from (1.8) by dividing both sides of (1.8) by Ø(s) provided that $\emptyset(s) \neq 0$.
In summary, the Fredholm Integral equation is of the first kind if the unknown function g(s) appears only under the integral sign. However, the Fredholm integral equation is of the second kind if the unknown function g(s) appears inside and outside the integral sign.

1.3.2 Volterra Linear Integral Equation

The standard form of Volterra linear integral equations, where the limits of integration are functions of **s** rather than constants, are of the form

$$\phi(s)g(s) = f(s) + \lambda \int_a^s K(s,t)\, g(t)dt \qquad (1.11)$$

Where the unknown function g(s) under the integral sign occurs linearly as stated before. It is worth noting that (1.11) can be viewed as a special case of the Fredholm integral equation when the kernel K(s, t) vanishes for t > s , s is in the range of integration [a, b]. As in Fredholm equations, Volterra integral equations fall under two kinds,

1. When Ø(x) = 0, equation (1.11) becomes

$$f(s) + \lambda \int_a^s K(s,t)\, g(t)dt \qquad (1.12)$$

And in this case the integral equation is called Volterra integral equation of the First Kind.

2. When Ø(s) = 1, equation (1.11) becomes

$$g(s) = f(s) + \lambda \int_a^s K(s,t)\, g(t)dt \qquad (1.13)$$

And in this case the integral equation is called the Volterra integral equation of the Second Kind. In summary, the Volterra integral equation is of the first kind if the unknown function g(s) appears only under the integral sign. However, the Volterra integral equation is of the second kind if the unknown function g(s) appears inside and outside the integral sign.

Examining the equations (1.8)-(1.13) carefully, the following remarks can be concluded.

Remarks

1. **The structure of Fredholm and Volterra equations**: The unknown function g(s) appears linearly only under the integral sign in linear Fredholm and Volterra integral equations of the First Kind. However, the unknown function g(s) appears linearly inside as well as outside the integral sign in second kind of both linear Fredholm and Volterra integral equations.

2. The Limits of Integration: In Fredholm integral equations, the integral is taken over a finite interval with fixed limits of integration. However, in Volterra integral equation, at least one limit of the range of integration is a variable, and the upper limit is the most commonly used with a variable limit.

3. The Linearity property: As indicated earlier, the unknown function g(s) in linear Fredholm and Volterra integral equations (1.10) and (1.13) occurs to the first power wherever it exists. However, nonlinear Fredholm and Volterra integral equations arise if g(s) is replaced by a nonlinear function F(g(s)) , such as $g^2(s)$,$e^{g(s)}$ and so on. The following are examples of nonlinear integral equations

$$g(s) = f(s) + \lambda \int_a^s K(s,t)\, g^2(t) dt. \qquad (1.14)$$

$$g(s) = f(s) + \lambda \int_a^s K(s,t)\, e^{g(t)} dt. \qquad (1.15)$$

$$g(s) = f(s) + \lambda \int_a^s K(s,t) \sin(g(t))\, dt. \qquad (1.16)$$

4. The Homogeneity property: On setting f (s) = 0 in Fredholm or Volterra integral equation of the second kind given by (1.10) and (1.13), the resulting equation is called a homogeneous integral equation, otherwise it is called nonhomogeneous integral equation.

1.3.3 Integro-Differential Equations

In this type of equations, the unknown function g(s) occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Further, we point out that an Integro-differential equation can be easily observed as an intermediate stage when we convert a differential equation to an integral equation.

The following are examples of Integro-differential equations:

$$g''(s) = -s + \int_0^s (s-t)\, g(t)dt\,, \quad g(0) = 0, g'(0) = 1. \tag{1.17}$$

$$g'(s) = -\sin s - 1 + \int_0^s g(t)\, dt, \quad g(0) = 1. \tag{1.18}$$

$$g'(s) = 1 - \frac{1}{3}s + \int_0^1 st\, g(t)dt, \quad g(0) = 1. \tag{1.19}$$

Equations (1.17), (1.18) are Volterra Integro-differential equations, and (1.19) is a FredholmIntegro-differential equation. This classification has been concluded as a result to the limit of integration.

1.3.4 Singular Integral Equations

The integral equation of first kind

$$f(s) = \lambda \int_{\alpha(s)}^{\beta(s)} K(s,t)\, g(t)dt \tag{1.20}$$

or the integral equation of second kind

$$g(s) = f(s) + \lambda \int_{\alpha(s)}^{\beta(s)} K(s,t)\, g(t)dt \tag{1.21}$$

is called singular if the lower limit, the upper limit or both limits of integration are infinite. In addition, the equation (1.20) or (1.21) is also called singular integral equation if the kernel K (s, t) becomes infinite at one or more points in the domain of integration. Examples of first type of singular integral equations are given by as

$$g(s) = 2s + 6\int_0^{\infty} \sin(s-t)\, g(t)dt \qquad (1.22)$$

$$g(s) = s + \frac{1}{3}\int_{-\infty}^{0} cos(s+t)\, g(t)dt \qquad (1.23)$$

$$g(s) = 1 + s^2 + \frac{1}{6}\int_{-\infty}^{\infty} (s+t)\, g(t)dt \qquad (1.24)$$

Where the singular behavior in these examples has resulted from the range of integration becoming infinite Examples of the second type of singular integral equations are given by

$$s^2 = \int_0^s \frac{1}{\sqrt{s-t}} g(t)dt. \qquad (1.25)$$

$$s = \int_0^s \frac{1}{(s-t)^{\alpha}} g(t)dt,\ 0 < \alpha < 1. \qquad (1.26)$$

$$g(s) = 1 - 2\sqrt{s} - \int_0^s \frac{1}{\sqrt{(s-t)}} g(t)dt. \qquad (1.27)$$

Where the singular behavior in this type of equations has resulted from the kernel K(s,t) becoming infinite as t → s. It is important to note that singular integral equations similar to examples (1.25) and (1.26) are called Abel's integral equation and generalized Abel's integral equation respectively. Singular integral equation similar to example (1.27) are called the weakly-singular second-kind Volterra type integral equations.

1.3.5 Volterra-Fredholm integral equations

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modeling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications. The standard form of the Volterra-Fredholm integral equation reads

$$g(s) = f(s) + \int_0^s K_1(s,t)\, g(t)dt + \int_a^b K_2(s,t)\, g(t)dt \qquad (1.28)$$

Where K_1 (s, t) and K_2 (s, t) are the kernels of the equation. Examples of the Volterra-Fredholm integral equations are

$$g(s) = 2s - \int_0^s (s-t)\, g(t)dt + \int_0^{\frac{\pi}{2}} s\, g(t)dt \qquad (1.29)$$

$$g(s) = \sin s - \cos s - \int_0^s g(t)\, dt + \int_0^{\frac{\pi}{2}} g(t)\, dt \qquad (1.30)$$

Notice that the unknown function g(s) appears inside the Volterra and Fredholm integrals and outside both integrals.

1.3.6 Volterra-FredholmIntegro Differential Equations:

The Volterra-Fredholm Differential Equation, which is a combination of disjoint Volterra and Fredholm integrals and Differential operator, may appear in one integral equation. The Volterra-FradholmIntegro-Differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm equations. The standard form of Volterra-FradholmIntegro-Differential equation reads

$$g^n(s) = f(s) + \int_0^s K_1(s,t)\, g(t)dt + \int_a^b K_2(s,t)\, g(t)dt \qquad (1.31)$$

Where K_1 (s, t) and K_2 (s, t) are the kernels of the equation, and n is the order of the ordinary derivative of g(s). Notice that because this kind of equations contains ordinary derivatives, then initial conditions should be prescribed depending on the order of the derivative involved. Examples of the Volterra-FredholmIntegro-differential equations are

$$g'(s) = 1 + \int_0^s (s-t)\, g(t)dt + \int_0^1 (st)\, g(t)dt\,, \qquad g(0) = 1 \qquad (1.32)$$

and

$$g''(s) = -s - \frac{1}{6}s^3 + \int_0^s g(t)dt + \int_{-\pi}^{\pi} (s)\, g(t)dt, \quad g(0) = 1, g'(0) = 2 \qquad (1.33)$$

Notice that the unknown function g(s) appears inside the Volterra and Fredholm integrals and outside both integrals. Finally in this section, we illustrate the classifications and the basic concepts that were discussed earlier by the following examples.

Example 1. Classify the following integral equation

$$g(s) = s - \frac{1}{6}s^3 + \int_0^s (s-t)\, g(t)dt \qquad (1.34)$$

asFredholm or Volterra integral equation, linear or nonlinear and homogeneous or nonhomogeneous. Note that the upper limit of the integral is and the function g(s) appears twice. This indicates that equation (1.34) is a Volterra integral equation of the second kind. The equation (1.34) is linear since the unknown function g(s) appears linearly inside and outside the integral sign. The presence of the function $f(s) = s - \frac{1}{6}s^3$ classifies the equation as a non-homogeneous equation.

Example 2. Classify the following integral equation

$$g(s) = \frac{1}{2} + s - \int_0^1 (s-t)\, g^2(t)dt \qquad (1.35)$$

As Fredholm or Volterra integral equation, linear or nonlinear and homogeneous or non-homogeneous. The limits of integration are constants and the function g(s) appears twice, therefore the equation (1.35) is a Fredholm integral equation of the second kind.

Further, the unknown function appears under the integral sign with power 2 indicating the equation is a non-linear equation. The non-homogeneous part f(s) appears in the equation showing that it is a non-homogeneous equation.

Check your Progress-1

1. Define Linear Integral Equation

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2. Explain Fredholm Linear Integral Equation

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3. State Volterra integral equation of the First Kind.

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1.4 RELATIONS BETWEEN DIFFERENTIAL AND INTEGRAL EQUATIONS

To convert the Differential Equations to Integral equations, the following results are necessary:

1.4.1 Leibnitz Rule of Differentiating Under The Integral Sign

If F(x,t) and $\frac{\partial F(x,t)}{\partial t}$ are continuous functions of x and t in the domain $\alpha \leq x \leq \beta$, $t_0 \leq t \leq t_1$,

then

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,t)dt$$

$$= \int_{a(x)}^{b(x)} \frac{\partial F(x,t)}{\partial x} dt + \frac{db(x)}{dx} F\big(x, b(x)\big) \qquad (1.36)$$

$$- \frac{da(x)}{dx} F(x, a(x))$$

provided the limits of integration a(x) and b(x) are defined functions having continuous derivatives for $\alpha \le x \le \beta$. This rule may be used to convert integral equations to equivalent ordinary differential equations. In particular, we have

(i) **For Volterra Integral Equation:**

$$\frac{d}{dx}\left[\int_a^x K(x,t)u(t)dt\right] = \int_a^x \frac{\partial K}{\partial x}\, u(t)dt + K(x,x)u(x) \qquad (1.37)$$

(ii) **For Fredholm Integral Equation:**

$$\frac{d}{dx}\left[\int_a^b K(x,t)u(t)dt\right] = \int_a^b \frac{\partial K}{\partial x}\, u(t)dt \qquad (1.38)$$

Here u(t) is independent of x and hence on taking partial derivatives with respect to x, u(t) is treated as constant.

1.4.2 Cauchy's Formula for Repeated Integration

Let f be a continuous function on real line. Then, the n^{th} repeated integral of f based at a is given by single integration:

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n)\, dx_n \dots dx_2 dx_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

Proof: The proof will be established using Mathematical Induction. Since f is continuous, the base case follows from the Fundamental theorem of calculus:

Let

$$I_n(x) = \int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n)\, dx_n \dots dx_2 dx_1$$

$$\frac{d}{dx}[I_1(x)] = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$I_1(a) = \int_a^a f(t)dt = 0$$

Now, suppose result is true for -, and let us prove it for - + 1. Apply the induction hypothesis and switching the order of integration,

$$I_{n+1}(x) = \int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_{n+1})\, dx_{n+1} \dots dx_2 dx_1$$

$$= \frac{1}{(n-1)!}\int_a^x \int_a^{x_1} (x_1 - t)^{n-1} f(t)dt\, dx_1$$

$$= \frac{1}{(n-1)!}\int_a^x \int_t^x (x_1 - t)^{n-1} f(t)dx_1 dt$$

[After changing the order of integration]

$$= \frac{1}{n!}\int_a^x (x-t)^n f(t)dt$$

Hence the proof follows. This n-fold integrals is an essential and useful formula that has enormous applications in the integral equation problems.

1.4.3 Converting IVP to Volterra Integral Equations

- **An Initial Value Problem**

Example 1

$$y''(x) = \lambda y(x) + g(x) \qquad (1.39)$$

$$y(0) = 1, y'(0) = 0 \qquad (1.40)$$

Let

$$y''(x) = F(x). \qquad (1.41)$$

Integrating with respect to between limit 0to x, we have

$$y' = \int_0^x F(t)dt + A$$

Again integrating

$$y'' = \int_0^x \int_0^x F(t)dt + A\int_0^x 1.dt + B$$

Where A and B are constant using Cauchy's Formula

$$y(x) = \int_0^x (x-t)F(t)dt + Ax + B \qquad (1.42)$$

Now using the initial conditions, y(0) = 1, y'(0) = 0 , we get

y(0) = 0 + B i.e., B=1

y'(0) = 0 + A ,i.e., A=0 Hence (1.42), becomes

$$y(x) = 1 + \int_0^x (x-t)F(t)dt \qquad (1.43)$$

From (1.39) and (1.42), we have

$$F(x) = \lambda y(x) + g(x) \qquad (1.44)$$

Now, substituting the value of F(x) from (1.44) in (1.43), we get

$$y(x) = 1 + \int_0^x (x-t)[\lambda y(t) + g(t)]dt$$

$$y(x) = 1 + \lambda \int_0^x (x-t)\, y(t)dt + \int_a^x (x-t)\, g(t)dt$$

which is a Volterra integral equation of second kind in y (x)

• **General Initial Value Problem** Reduce the initial value problem

$$y''(x) + u(x)y'(x) + v(x)y(x) = g(x)$$

$$y(a) = A, y'(a) = B$$

Solution:

$$y''(x) = -u(x)y'(x) - v(x)y(x) + g(x) \tag{1.45}$$

Integrating over ‘a’ to ‘x’, we have

$$[y'(x) - y'(a)] = -\int_a^x u(t)y'(t)dt - \int_a^x v(t)y(t)dt + \int_a^x g(t)dt$$

Integrating by parts, we have

$$y'(x) - B = -[u(t)y(t)]|_a^x + \int_a^x u'(t)y(t)dt - \int_a^x v(t)y(t)dt + \int_a^x g(t)dt$$

or

$$y'(x) - B = \int_a^x [u'(t) - v(t)]y(t)dt + \int_a^x g(t)dt - u(x)y(x) + u(a)A$$

Again integrating over a to x, we obtain by Cauchy’s Formula

$$[y(x) - y(a)] - B(x-a) = \int_a^x (x-t)[u'(t) - v(t)]y(t)dt + \int_a^x (x-t)g(t)dt$$

$$- \int_a^x u(t)y(t)dt + \text{u(a) A (x-a)}$$

It implies

$$y(x) = F(x) + \int_a^x K(x,t)y(t)dt$$

where

$$F(x) = A + B(x-a) + \int_a^x (x-t)g(t)dt + \text{u(a) A (x} - \text{a)}$$

$$K(x,t) = (x-t)[u'(t) - v(t)] - u(t)$$

this is required result.

Note: To convert an IVP to Volterra Integral equation, integrate between the initial value and x.

1.4.4 Converting BVP to Fredholm Integral Equations

The method is similar to that discussed in previous section with some exceptions that are related to the boundary conditions. We demonstrate this method with an illustration.

Check your Progress-2

4. State Cauchy's Formula for Repeated Integration

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5. Explain how to convert IVP to Volterra Integral Equations

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1.5LET US SUM UP

In this chapter, we have discussed the classifications of integral equations such as Fredholm Integral Equation, Volterra Integral Equation and many others along with the kinds of particular equation. After that we studied the relation between Integral equation and Differential equation with the conversion of Initial Value Problem into Volterra Integral Equation and Boundary Value Problem into Fredholm Integral Equation.

1.6 KEYWORDS

1. **Dimensional Vector:** In mathematics, the dimension of a vector space **V is the** cardinality (i.e. the number of vectors) **of** a basis of V over **its** base field
2. **Finite Interval** - **A** finite **interval** (bounded **interval**) is **an interval**, whose both endpoints are numbers

1.7QUESTIONS FOR REVIEW

1. Classify each of the following integral equations as Volterra or Fredholm integral equation, linear or nonlinear, and homogeneous or non-homogeneous;

a) $u(x) = x + \int_0^1 (x-t)^2 u(t)dt$

b) $u(x) = e^x + \int_0^x t^2 u^2(t)dt$

Find the equivalent Volterra integral equation to the following initial value problem

$$y''(x) + y(x) = \cos x \quad , \qquad y(0) = 0 \ , \ \ y'(0) = 1.$$

1.8 SUGGESTED READINGS AND REFERENCES

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1.9 ANSWERS TO CHECK YOUR PROGRESS

1. Provide definition – 1.2

2. Provide definition with the help of equation – 1.3.1

3.Provide explanation with the help of equation – 1.3.2

4. Provide Formula – 1.4.2

4. Provide explanation – 1.4.3

UNIT-2 KERNELS AND ITS TYPES

STRUCTURE

2.0 OBJECTIVES

Understand the concept of Different types of kernels

Understand Eigenvalues and Eigen functions

Enumerate Reduction of Integral Equations to System of Algebraic Equations

2.1 INTRODUCTION

We will explore different types of kernels in this Chapter. We will understand the relationship of Eigen values and functions with kernels.

2.2 TYPES OF KERNELS

The following special cases of the kernel of an integral equation are of main interest:-

(i) Symmetric Kernel

(ii) Separable Kernel

(iii) Resolvent Kernel

(iv) Iterated Kernels

2.2.1 Symmetric Kernel

A Kernel K (s,t) is symmetric (or complex symmetric or Hermitian) if

$$K(s,t) = K^*(t, s)$$

where the asterisk denotes the complex conjugate. For a real kernel, this coincides with definition

$$K(s,t) = K(t, s)$$

For example, sin(x,t) , log xt , $x^2t^2 + xt + 1$ etc. are all symmetric kernels. Again, sin(2x + 3t) and $x^2t^3 + 1$ are not symmetric kernels. Again i(x − t) is a symmetric kernel, since in this case, if K (x, t) = i(x – t), then K (t, x) = i (t – x), and so

$$\overline{K}(t,x) = -i(t-x) = i(x-t) = K(x,t).$$

On the other hand, i(x + t) is not a symmetric kernel, since in this case, if K (x, t) = i(x + t), then

$$\overline{K}(t,x) = \overline{i(t+x)} = -i(t+x) = -K(x,t)$$

and so

$$K(x,t) \neq \overline{K}(x,t).$$

2.2.2 Separable or Degenerate Kernel

A kernel K(s,t) is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of s only and a function of t only, that is

$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(s)$$

Remark. The functions $a_i(s)$ can be assumed to be linearly independent, otherwise the number of terms in relation (1.59) can be reduced (by linear independence it is meant that, if $c_1a_1 + c_2a_2 + \ldots + c_na_n = 0$. where c_i are arbitrary constants, then $c_1 = c_2 = \ldots = c_n = 0$

2.2.3 Resolvent Kernel Suppose solution of integral equations

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt$$

and

$$y(x) = f(x) + \lambda \int_a^x K(x,t) y(t) dt$$

be respectively

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) y(t) dt$$

and

$$y(x) = f(x) + \lambda \int_a^x \Gamma(x,t;\lambda) y(t) dt$$

Then R (x, t ; λ)or Γ(x, t ; λ)is called the Resolvent kernel or reciprocal kernel of the given integral equation

2.2.4 Iterated Kernels

Consider Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

Then, the iterated kernels $K_n(x,t)$, n = 1,2,3, ⋯ are defined as follows:

$$K_1(x,t) = K(x,t)$$

$$K_n(x,t) = \int_a^b K(x,z)K_{n-1}(x,t)dt \quad n = 2,3,\ldots$$

2.3 EIGENVALUES AND EIGEN FUNCTIONS

Consider the homogenous Fredholm Integral equation

$$y(x) = \lambda \int_a^b K(x,t)y(t)dt \qquad (2.0)$$

Then (2.0) has the obvious solution y(x) = 0, which is called the zero or trivial solution of (2.0). The values of the parameter λ for which (2.0) has a non-zero solution y(x) $\neq 0$ are called the eigenvalues of (2.0) or of the kernel K(s, t) and every non-zero solution of (2.0) is called an eigenfunction corresponding to the eigenvalue λ.

Remark 1. The number λ = 0 is not an eigenvalue since for λ = 0 it follows from (2.0) that y(x) = 0.

Remark 2. If y(x) is an eigenfunction of (2.0), then cy(x), where c is an arbitrary constant, is also an eigen function of (2.0) which corresponds to the same eigenvalue λ.

Remark 3.A homogenous Fredholm integral equation of the second kind may, generally, have no eigenvalue and eigenfunction, or it may not have any real eigenvalue or eigenfunction**.**

Check your Progress-1

1. Define

a. Separable or Degenerate Kernel

b. 2.2.4 Iterated Kernels

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2. What is Eigen Value and Eigen Function?

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2.4 REDUCTION OF INTEGRAL EQUATIONS TO SYSTEM OF ALGEBRAIC EQUATIONS

In Lesson-l, we have defined a degenerate or a separable kernel K (s, t) as

$$K(s,t) = \sum_{i=1}^{n} a_i(s)b_i(t) \tag{2.1}$$

where the functions $a_1(s)$, $a_2(s)$, … $a_n(s)$ and the functions $b_1(s)$, $b_2(s)$, … $b_n(s)$ are linearly independent. With such a kernel, the Fredholm integral equation of the second kind.

$$g(s) = f(s) + \lambda \int K(s,t)\, g(t)dt \qquad (2.2)$$

becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t) g(t)dt \qquad (2.3)$$

It emerges that the technique of solving this equation is essentially dependent on the choice of the complex parameter ' and on the definition of

$$c_i = \int b_i(t) g(t)dt \qquad (2.4)$$

The quantities c_i are constants. Substituting Equations (2.4) in (2.3) we get

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) \qquad (2.5)$$

and the problem reduces of finding the quantities c_i. To this end, we put the value of g(s) as given by equation (2.5) in (2.3) and get

$$\sum_{i=1}^{n} a_i(s) \left[c_i - \int b_i(t) \left[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right] dt \right] = 0 \qquad (2.6)$$

but the functions $a_i(s)$ are linearly independent, therefore,

$$c_i - \int b_i(t) \left[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right] dt = 0\,, \quad i = 1,2,\cdots,n \qquad (2.7)$$

Using the simplified notation

$$\int b_i(t) f(t) dt = f_i\,, \qquad \int b_i(t) a_k(t) dt = a_{ik}. \qquad (2.8)$$

Where f_i and a_{ik} are known constants, equation (2.7) becomes

$$c_i - \lambda \sum_{k=1}^{n} c_k a_{ik} = f_i\,, \quad i = 1,2,\cdots,n \tag{2.9}$$

that is, a system of n algebraic equations for the unknowns c_i. The

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \vdots & & & \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \tag{2.10}$$

which is a polynomial in λ of degree at most n. Moreover, it is not identically zero. Since, when λ = 0, it reduces to unity. For all values of λ for which D(λ) ≠ 0, the algebraic system (2.9), and thereby the integral equation (2.2), has a unique solution. These values of λ are called regular, on the other hand, for all values of λ for which D(λ) becomes equal to zero, the algebraic system (2.9) and with it the integral equation (2.2), either insoluble or has an infinite number of solutions. Setting λ = 1/μ in equation (2.9) we have the eigenvalue problem of matrix theory. The eigenvalues are given by the polynomial D(λ) = 0.They are also the eigenvalues of integral equation.

Example 1. Solve the Fredholm integral equation of second kind

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t)\, g(t)dt \tag{2.11}$$

$$c_1 = \int_0^1 t^2\, g(t)dt\,, \quad c_2 = \int_0^1 t\, g(t)dt \tag{2.12}$$

therefore equation (2.11) becomes

$$g(s) = s + \lambda c_1 s + \lambda c_2 s^2\ , \tag{2.13}$$

Using (2.12) in (2.11), we obtain the algebraic equations

$$c_1 = \frac{1}{4} + \frac{1}{4}\lambda c_1 + \frac{1}{5}\lambda c_2$$

$$c_2 = \frac{1}{3} + \frac{1}{3}\lambda c_1 + \frac{1}{4}\lambda c_2 \qquad (2.14)$$

The solution of these equations is readily obtained as

By equations (2.13) and (2.15), we get the solution

$$g(s) = \frac{[(240 - 60\lambda)s + 80\lambda s^2]}{240 - 120\lambda - \lambda^2}\,, \qquad (2.16)$$

$$c_1 = \frac{60 + \lambda}{240 - 120\lambda - \lambda^2}\,, \quad c_2 = \frac{80}{240 - 120\lambda - \lambda^2} \qquad (2.15)$$

Example 2. Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (s + t)g(t)dt \qquad (2.17)$$

And find the eigenvalues.

Here, $a_1(s) = s, a_2(s) = 1\,, b_1(t) = 1, b_2(t) = t$

$$a_{11} = \int_0^1 t dt = \frac{1}{2} \quad , \quad a_{12} = \int_0^1 dt = 1\ ,$$

$$a_{21} = \int_0^1 t^2 dt = \frac{1}{3} \quad , \quad a_{22} = \int_0^1 t dt = 1/2\ ,$$

$$f_1 = \int_0^1 f(t)dt, \quad f_2 = \int_0^1 t f(t)dt$$

Substituting these values in equation (2.9), we have the algebraic system

$$\left(1-\frac{1}{2}\lambda\right)c_1-\lambda c_2=f_1 \quad , \qquad -\frac{1}{3}\lambda c_1+\left(1-\frac{1}{2}\lambda\right)c_2=f_2,$$

The determinant D(λ) = 0 gives $\lambda^2 + 12\lambda - 12 = 0$. Thus, the eigenvalues are

$$\lambda_1=\left(-6+4\sqrt{3}\right), \qquad \lambda_2=\left(-6-4\sqrt{3}\right),$$

For these two values of λ., the homogenous equation has a nontrivial solution, whereas the integral equation (2.17) is, in general, not soluble. When λ differs from these values, the solution of the preceding algebraic system is

$$c_1=\frac{-12f_1+\lambda(6f_1-12f_2)}{\lambda^2+12\lambda-12}$$

$$c_2=\frac{-12f_2+\lambda(4f_1-6f_2)}{\lambda^2+12\lambda-12}$$

Using the relation (2.5), there results the solution

$$g(s)=f(s)+\lambda\int_0^1\frac{6(\lambda-2)(s+t)-12\lambda st-4\lambda}{\lambda^2+12\lambda-12}f(t)dt \qquad (2.18)$$

The function Γ(s, t; λ),

$$\Gamma(s,t;\lambda)=\frac{6(\lambda-2)(s+t)-12\lambda st-4\lambda}{\lambda^2+12\lambda-12} \qquad (2.19)$$

is called the resolvent kernel. We have. therefore succeeded in inverting the integral equation because the right-hand side of the preceding formula is a known quantity.

Example 3. Invert the integral equation

$$\Gamma(s,t;\lambda) = \frac{6(\lambda-2)(s+t) - 12\lambda st - 4\lambda}{\lambda^2 + 12\lambda - 12} \qquad (2.19)$$

As in the previous examples, we set

$$c = \int_0^{2\pi} (\cos t) g(t) dt$$

To obtain

$$g(s) = f(s) + \lambda c \sin s\,. \qquad (2.21)$$

Multiply both sides of this equation by cos and integrate from 0to 2ê.
This gives

$$c = \int_0^{2\pi} (\cos t) f(t) dt \qquad (2.22)$$

From equations (2.22) and (2.21), we have the required formula:

$$g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t)\, f(t) dt. \qquad (2.23)$$

Example 4. Find the resolvent kernel for the integral equation

$$g(s) = f(s) + \lambda \int_{-1}^{1} (st + s^2 t^2)\, g(t) dt. \qquad (2.24)$$

For this equation,

$$a_1(s) = s\ , a_2(s) = s^2\ , b_1(t) = t\ , b_2(t) = t^2\ ,$$

$$a_{11} = \frac{2}{3}, \;\; a_{12} = a_{21} = 0\,, \;\; a_{22} = \frac{2}{3},$$

$$f_1 = \int_{-1}^{1} t f(t) dt\,, \;\; f_2 = \int_{-1}^{1} t^2 f(t) dt$$

Therefore, the corresponding algebraic system is

$$c_1\left(1 - \frac{2}{3}\lambda\right) = f_1\,, \quad c_2\left(1 - \frac{2}{3}\lambda\right) = f_2 \qquad (2.25)$$

Substituting the values of c_1 and c_2 as obtained from Equation (2.25) in (2.5) yields the solution

$$g(s) = f(s) + \lambda \int_{1}^{1} \left[\frac{st}{1 - \frac{2}{3}\lambda} + \frac{s^2t^2}{1 - \frac{2}{3}\lambda}\right] f(t) dt \qquad (2.26)$$

Thus, the resolvent kernel is

$$\Gamma(s, t; \lambda) = \frac{st}{1 - \frac{2}{3}\lambda} + \frac{s^2t^2}{1 - \frac{2}{3}\lambda} \qquad (2.27)$$

2.5 FREDHOLM ALTERNATIVE

In the previous sections, we have seen that, if the kernel is separable, the problem of solving an integral equation of the second kind reduces to that of solving an algebraic system of equations. Although the integral equations with degenerate kernels are not found frequently in practice, yet the results derived for such equations are essential to study integral equations of more general types.

Furthermore, any reasonably well-behaved kernel can be expressed as an infinite series of degenerate kernels. When an integral equation cannot be solved in closed form, then we have to use approximate methods to solve a given integral equation. However any approximate methods can be employed with confidence only if the existence of the solution is known in advance.

The Fredholm theorems explained in this Lesson plan provide such an assurance. The basic theorems of the general theory of integral equations, which were first presented by Fredholm, correspond to the basic theorems of linear algebraic systems. Here, we shall deal with degenerate kernels and borrow the results of linear algebra. In Section 2.3, we have found that the solution of the present problem rests on the investigation of the determinant (2.10) of the coefficients of the algebraic system (2.9). If $D(\lambda) \neq 0$, then that system has only one solution, given by Cramer's rule

$$c_i = \frac{(D_{1i}f_1 + D_{2i}f_2 + \cdots + D_{ni}f_n)}{D(\lambda)}, \quad i = 1,2,\cdots,n \qquad (2.28)$$

Where D_{hi} denotes the cofactor of the (*h*, s)th element of the determinant (2.10). Consequently, the integral equation (2.2) has the unique solution (2.5), which, in view of (2.33), becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \frac{(D_{1i}f_1 + D_{2i}f_2 + \cdots + D_{ni}f_n)}{D(\lambda)} a_i(s) \qquad (2.29)$$

while the corresponding homogeneous equation

$$g(s) = \lambda \int K(s,t)g(t)dt \qquad (2.30)$$

has only the trivial solution g(s) = 0. Substituting for f_i from (2.8) in (2.29), we can write the solution g(s) as

$$g(s) = f(s) + \left[\frac{\lambda}{D(\lambda)}\right] \tag{2.31}$$

$$\times \int \left\{\sum_{i=1}^{n} [D_{1i}b_1(t) + D_{2i}b_2(t) + \cdots + D_{ni}b_n(t)]a_i(s)\right\} f(t)dt$$

Now consider the determinant of (n + 1)th order

$$D(s,t;\lambda) = \begin{vmatrix} 0 & a_1(s) & \cdots & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & \cdots & -\lambda a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n(t) & -\lambda a_{n1} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \tag{2.32}$$

By developing it by the elements of the first row and the corresponding minors by the elements of the first column, we find that the expression in the brackets is D(s, t ; λ) With the definition

$$\Gamma(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)} \tag{2.33}$$

Equation (2.30) takes the simple form

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)dt \tag{2.34}$$

The function Γ(s, t ; λ) is the resolvent (or reciprocal) kernel we have already encountered in Examples 2 and 4 in the previous section. For the time being, we content ourselves with the observation that the only possible singular points of Γ(s, t ; λ) in the λ plane are the roots of the equation D(λ) = 0, i.e., the eigen values of the kernel

K(s,t) The above discussion leads to the following basic Fredholm theorem

Check your Progress-2

3.State the necessary remarks for Eigen value and Eigen Function

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4. Explain Fredholm Alternative

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2.6 LET US SUM UP

In this chapter, we have discussed separable kernel and how to solve Fredholm integral equation of the second kind with separable kernel. Also we discussed the method of finding eigenvalue and Eigen function of the fredholm integral equation of the second kind by reducing the equation to an algebraic system of equation.

2.7 KEYWORDS

3. **Linearly independent function :** Two functions y_1 and y_2 are **said** to **be** linearly independent **if** neither function **is** a constant multiple **of** the other.
4. **Polynomial -** an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s).

3. Insoluble - infinite number of solutions

2.8 QUESTIONS FOR REVIEW

1. Solve the homogenous Fredholm integral equation

2. Find the eigenvalues and

$$g(s) = \lambda \int_0^1 e^s e^t g(t) dt$$

eigenfunctions of the homogenous integral equation

$$g(s) = \lambda \int_1^2 \left[st + \frac{1}{st}\right] g(t)dt$$

3. Solve the following integral equations:

(i) $y(x) - \lambda \int_0^{2\pi} |\pi - t| \sin x \; y(t)dt = x$

(ii) $y(x) = x + \lambda \int_0^1 (1 + x + t) y(t)dt$

2.9 SUGGESTED READINGS AND REFERENCES

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2. Linear Integral Equation: W.V. Lovitt (Dover).
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4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
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6. D. Powers, Boundary Value Problems Academic Press, 1979.

2.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide definition – 2.2.2 & 2.2.4

2. Provide explanation– 2.3

3. Provide remarks – 2.3

4. Provide explanation – 2.5

UNIT-3 FREDHOLM ALTERNATIVE THEOREM

STRUCTURE

3.0 OBJECTIVES

Comprehend the Fredholm Alternative Theorem and An Approximate Method

Understand the concept of Iterated Kernels or Functions

Understand the concept of Resolvent Kernel or Reciprocal Kernel

3.1 INTRODUCTION

In mathematics, **Fredholm's theorems** are a set of celebrated results of Ivar Fredholm in the Fredholm theory of integral equations. There are several closely related theorems, which may be stated in terms of integral equations, in terms of linear algebra, or in terms of the Fredholm operator on Banach spaces.

The Fredholm alternative is one of the Fredholm theorems.

3.2 FREDHOLM ALTERNATIVE THEOREM.

Either the integral equation

$$g(s) = f(s) + \lambda \int K(s,t)g(t)dt \qquad (3.0)$$

with fixed λ possesses one and only one solution g(s) for arbitrary functions f(s) and K(s,t), in particular the solution g = 0 for f = 0; or the homogeneous equation

$$g(s) = \lambda \int K(s,t)g(t)dt \qquad (3.1)$$

possesses a finite number r of linearly independent solutions g_{0_i}, i = 1,2, ⋯ , r. In the first case, the transposed inhomogeneous equation

$$\psi(s) = f(s) + \lambda \int K(t,s)\psi(t)dt \qquad (3.2)$$

also possesses a unique solution. In the second case, the transposed homogeneous equation

$$\psi(s) = \lambda \int K(t,s)\psi(t)dt \qquad (3.3)$$

also has r linearly independent solutions ψ_{0_i}, s = 1,2, ⋯ , r; the inhomogeneous integral equation (2.27) has a solution if and only if the given function f(s) satisfies the r conditions

$$(f, \psi_{0_i}) = \int f(s)\psi_{0_i}(s)ds = 0 \qquad (3.4)$$

In this case, the solution of (3.1) is determined only up to an additive linear combination $\sum_{i=1}^{r} c_i g_{0_i}$

The following examples illustrate the theorems of this section.

Example 1. Show that the integral equation

$$g(s) = f(s) + \left(\frac{1}{\pi}\right)\int_0^{2\pi} [\sin(s+t)]g(t)dt \qquad (1)$$

possesses no solution for f(s) = s, but that it possesses infinitely many solutions when f(s) = 1. For this equation,

$$K(s,t) = \sin s \cos t + \cos s \sin t,$$

$$a_1(s) = \sin s\ , \quad a_2(s) = \cos s\ , \quad b_1(t) = \cos t\ , \quad b_2(t) = \sin t.$$

Therefore,

$$a_{11} = \int_0^{2\pi} \sin t \cos t\, dt = 0 = a_{22}\ ,$$

$$a_{12} = \int_0^{2\pi} \cos^2 t\, dt = \pi = a_{21}\ .$$

$$D(\lambda) = \begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 1 - \lambda^2\pi^2 \qquad (2)$$

The eigenvalues are

$$\lambda_1 = \frac{1}{\pi}\ , \lambda_2 = -\frac{1}{\pi}$$

and equation (1) contains $\lambda_1 = 1/\pi$

Therefore, we have to examine the Eigen functions of the transposed equation (note that the kernel is symmetric)

$$g(s) = \left(\frac{1}{\pi}\right) \int_0^{2\pi} \sin(s+t)\, g(t) dt \qquad (3)$$

The algebraic system corresponding to (3) is

$$c_1 - \lambda \pi c_2 = 0, \qquad -\lambda \pi c_1 + c_2 = 0$$

which gives

$$c_1 = c_2 \text{for} \quad \lambda_1 = \frac{1}{\pi}\,; \qquad c_1 = -c_2 \text{for} \lambda_2 = -\frac{1}{\pi}$$

Therefore, the Eigen functions for $\lambda_1 = 1/\pi$ I follow from the relation (2.5) and are given by

$$g(s) = c(\sin s + \cos s) \qquad (3)$$

Since

$$\int_0^{2\pi} (s \sin s + s \cos s) ds = -2\pi \neq 0$$

While

$$\int_0^{2\pi} (\sin s + \cos s) ds = 0$$

Example 2. Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (1 - 3st) g(t) dt \qquad (1)$$

The algebraic system (2.9) for this equation is

$$(1-\lambda)c_1 + \frac{3}{2}\lambda c_2 = f_1 \,, \qquad -\frac{1}{2}\lambda c_1 + (1+\lambda)c_2 = f_2 \tag{2}$$

While

$$D(\lambda) = \begin{vmatrix} 1-\lambda & \frac{3}{2}\lambda \\ -\frac{1}{2}\lambda & 1+\lambda \end{vmatrix} \tag{3}$$

Therefore, the inhomogeneous equation (1) will have a unique solution if and only if $\lambda \neq \pm 2$. Then the homogeneous equation

$$g(s) = \lambda \int_0^1 (1-3st)g(t)dt \tag{4}$$

has only the trivial solution. Let us now consider the case when ' is equal to one of the eigen values and examine the eigen functions of the transposed homogeneous equation

$$g(s) = \lambda \int_0^1 (1-3st)g(t)dt \tag{5}$$

For $\lambda = +2$, the algebraic system (2.59) gives $c_1 = 3c_2$. Then, (2.5) gives the eigenfunction

$$g(s) = c(1-s) \tag{6}$$

where c is an arbitrary constant. Similarly, for $\lambda = -2$, the corresponding Eigen function is

$$g(s) = c(1-3s) \tag{7}$$

It follows from the above analysis that the integral equation

$$g(s) = f(s) + 2\int_0^1 (1-3st)g(t)dt$$

will have a solution if f(s) satisfies the condition

$$\int_0^1 (1-s)\, f(s) ds = 0$$

while the integral equation

$$g(s) = f(s) - 2\int_0^1 (1-3st) g(t) dt$$

Will have a solution if the following holds:

$$\int_0^1 (1-3s) f(s) ds = 0$$

3.3 AN APPROXIMATE METHOD

We propose to describe a useful method for finding approximate solutions of some special type of integral equations. We shall explain this method with the help of following example

Example 1. Solve the integral equation

$$\mathrm{y(x) = e^x - x - \int_0^1 x(e^{xt} - 1)y(t)dt}$$

Solution: Let us approximate the kernel by the sum of the first three terms in its Taylor series :

$$\mathrm{K(x,t) = x(e^{xt} - 1) \simeq x\left\{1 + \frac{xt}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} - 1\right\}}$$

Which is a separable kernel. Then, the given integral equation takes the form

$$y(x) = e^x - x - \int_0^1 \left(x^2t + \frac{1}{2}x^3t^2 + \frac{1}{6}x^4t^3\right) y(t)dt \qquad (1)$$

$$C_2 = -\frac{1}{2}\int_0^1 t^2 y(t)dt \qquad (4)$$

$$C_3 = -\frac{1}{6}\int_0^1 t^3\, y(t)dt \qquad (5)$$

Then (1) gives

$$y(x) = e^x - x + C_1x^2 + C_2x^3 + C_3x^4 \qquad (6)$$

$$y(t) = e^t - t + C_1t^2 + C_2t^3 + C_3t^4 \qquad (7)$$

Substituting the value of y(t) given by (2.71) in (2.67) we get

$$C_1 = -\int_0^1 t(e^t - t + C_1t^2 + C_2t^3 + C_4t^4)dt$$

$$C_1 = -[te^t - e^t]_0^1 + \frac{1}{3} - \left(\frac{C_1}{4}\right) - \left(\frac{C_2}{5}\right) - \left(\frac{C_3}{6}\right) = -\frac{2}{3}$$

Substituting the value of y(t) given by (7) in (4), we get

$$C_2 = -\frac{1}{2}\int_0^1 t(e^t - t + C_1t^2 + C_2t^3 + C_4t^4)dt$$

$$C_2 = -\frac{1}{2}[t^2e^t - 2te^t + 2e\text{^}t]_0^1 + \frac{1}{8} - \left(\frac{C_1}{10}\right) - \left(\frac{C_2}{12}\right) - \left(\frac{C_3}{14}\right)$$

$$\frac{C_1}{10} + \left(\frac{13}{12}\right) \times C_2 + \frac{C_3}{14} = -\left(\frac{1}{2}\right) \times (e - 2e + 2e - 2) + \frac{1}{8}$$

$$\frac{1}{5}C_1 + \frac{13}{6}C_2 + \frac{1}{7}C_2 = \frac{9}{4} - e \qquad (8)$$

Substituting the value of y(t) given by (7) in (5), we get

$$C_3 = -\frac{1}{6}\int_0^1 t^3(e^t - t + C_1t^2 + C_2t^3 + C_3t^4)dt$$

$$C_3 = -\frac{1}{6}\int_0^1 t^3 e^t dt + \frac{1}{6}\int_0^1 t^4 dt - \frac{C_1}{6}\int_0^1 t^5 dt - \frac{C_2}{6}\int_0^1 t^6 dt - \frac{C_3}{6}\int_0^1 t^7 dt$$

$$C_3 = -\frac{1}{6}[(t^3)e^t - 3t^2e^t + 6te^t - 6e^t]_0^1 + \frac{1}{30} - \frac{1}{36}C_1 - \frac{1}{42}C_2 - \frac{1}{48}C_3$$

$$\frac{1}{6}C_1 + \frac{1}{7}C_2 + \frac{49}{8}C_2 = 2e - \frac{29}{5} \qquad (9)$$

Solving (7), (8) and (9) leads to

$$C_1 = -0.5010, C_2 = -0.1671 \; and \; C_3 = -0.0422$$

With these values, (6) gives the required approximate solution of (1) as

$$y(x) = e^x - x - 0.5010x^2 - 0.1671x^3 - 0.0422x^4 \qquad (10)$$

Now as usual, we prove that the exact solution of given equation

$$y(x) = e^x - x - \int_0^1 x(e^{xt} - 1)y(t)dt \qquad (11)$$

is given by

$$y(x) = 1 \qquad (12)$$

From (12), y(t) = 1. Then, we have

$$\text{R.H.S of (2.75)} = e^x - x - \int_0^1 x(e^{xt} - 1)dt = e^x - x - x\left[\frac{e^{xt}}{x}\right]_0^1 + [xt]_0^1 = e^x - x - (e^x - 1) + x = 1 = y(x) = \text{L.H.S. of } (11)$$

Hence y(x) = 1 is the exact solution of (12).

Using the approximate solution (11) for x = 0, x = 0.5and x = 1.0, the values of y(x)=1.0000, y(0.5) = 1.0000 and y(1)=1.0080 which agrees with exact solution (12) rather closely.

3.4 ITERATED KERNELS OR FUNCTIONS

Definition.

i) Consider Fredholm integral equation of the second kind

Then, the iterated K_n : n = 1, 2, 3, … are defined as follows:

$$K_1(x,t) = K(x,t) \qquad (2)$$

and

$$\left.\begin{array}{l} K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz,\ n = 2,3,\ldots \\ \text{or} \\ K_n(x,t) = \int_a^b K_{n-1}(x,z)K(z,t)dz,\ n = 2,3,\ldots \end{array}\right\} \qquad (3)$$

$$y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt \qquad (4)$$

(ii) Consider Volterra integral equation of the second kind

Then, the iterated K_n : n = 1, 2, 3, … are defined as follows:

$$K_1(x,t) = K(x,t) \qquad (5)$$

And

$$\left.\begin{array}{l} K_n(x,t) = \int_a^x K(x,z)K_{n-1}(z,t)dz,\ n = 2,3,\ldots \\ \text{or} \\ K_n(x,t) = \int_a^x K_{n-1}(x,z)K(z,t)dz,\ n = 2,3,\ldots \end{array}\right\} \qquad (6)$$

3.5 RESOLVENT KERNEL OR RECIPROCAL KERNEL

Suppose solution of Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \qquad (1)$$

or $$\left.\begin{aligned} y(x) &= f(x) + \lambda \textstyle\int_a^b R(x,t;\lambda)f(t)dt \\ y(x) &= f(x) + \lambda \textstyle\int_a^b \Gamma(x,t;\lambda)f(t)dt \end{aligned}\right\} \qquad (2)$$

Then R(x, t; λ) or Γ(x, t; λ) is known as the resolvent kernel of (1).

ii) Suppose solution of Volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt \qquad (3)$$

or $$\left.\begin{aligned} y(x) &= f(x) + \lambda \textstyle\int_a^x R(x,t;\lambda)f(t)dt \\ y(x) &= f(x) + \lambda \textstyle\int_a^x \Gamma(x,t;\lambda)f(t)dt \end{aligned}\right\} \qquad (4)$$

Then R(x, t; λ) or Γ(x, t; λ) is known as the resolvent kernel of (3).

3.5.1 Theorem: The m^{th} iterated kernel Km(x, t) satisfies the relation

$$K_m(x,t) = \int_a^b K_r(x,y)K_{m-r}(y,t)dt$$

where r is any positive integer less than m.

Proof. The m^{th} iterated kernel $K_m(x, t)$ is defined as

$$K_1(x,t) = K(x,t) \qquad (5)$$

and

$$K_m(x,t) = \int_a^b K(x,s)K_{m-1}(s,t)ds, m = 2, 3, \ldots \qquad (6)$$

Re-writing (6), we have

$$K_m(x,t) = \int_a^b K(x,s_1)K_{m-1}(s_1,t)ds_1 \qquad (7)$$

Replacing m by m – 1 in (6), we have

$$K_{m-1}(x,t) = \int_a^b K(x,s)K_{m-2}(s,t)ds = \int_a^b K(x,s_2)K_{m-2}(s_2,t)ds_2$$

or

$$K_{m-1}(x,s_1) = \int_a^b K(s_1,s_2)K_{m-2}(s_2,t)ds_2 \qquad (8)$$

Using (8), (7) reduces to

$$K_m(x,t) = \int_a^b K(x,s_1)\left\{\int_a^b K(s_1,s_2)K_{m-2}(s_2,t)ds_2\right\}ds_1$$

$$K_m(x,t) = \int_a^b \int_a^b K(x,s_1)K(s_1,s_2)K_{m-2}(s_2,t)ds_2\, ds_1$$

Proceeding likewise, we obtain

$$K_m(x,t) = \int_a^b \int_a^b \dots \int_a^b K(x,s_1)K(s_1,s_2)K(s_2,s_3)\dots K_1(s_{m-1},t)ds_{m-1}\dots ds_2\, ds_1$$

or $K_m(x,t)$

$$= \underbrace{\int_a^b \int_a^b \dots \int_a^b}_{\substack{(m-1)th\ order \\ integral}} K(x,s_1)K(s_1,s_2)\dots K(s_{r-1},s_r)K(s_r,s_{r+1})\dots \dots K_1(s_{m-1},t)ds_{m-1}\dots ds_2 ds_1$$

(9)

Note that R.H.S. of (9) is a multiple integral of order + – 1.Proceeding as above, we may also write.

$$K_r(x,y) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{\substack{(r-1)th\ order \\ integral}} K(x,u_1) \ldots K(u_1,u_2) \ldots K(u_{r-1},y)du_{r-1} \ldots du_2 du_1 \qquad (10)$$

and

$$K_{m-r}(y,t) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{\substack{(m-r-1)th\ order \\ integral}} K(y,v_1) \ldots K(v_1,v_2) \ldots K(v_{m-r-1},t)dv_{m-r-1} \ldots dv_2 dv_1 \qquad (11)$$

Now, $\int_a^b K_r(x,y)K_{m-r}(y,t)dy =$

$$\int_a^b \left[\left\{\int_a^b \int_a^b \cdots \int_a^b K(x,u_1) \ldots K(u_1,u_2) \ldots K(u_{r-1},y)du_{r-1} \ldots du_2 du_1\right\} \times \right.$$

$$\left.\left\{\int_a^b \int_a^b \cdots \int_a^b K(y,v_1) \ldots K(v_1,v_2) \ldots K(v_{m-r-1},t)dv_{m-r-1} \ldots dv_2 dv_1\right\}\right]dy,$$

[by (10) and (11)]

or $\int_a^b K_r(x,y)K_{m-r}(y,t)dy$

$$= \int_a^b \int_a^b \cdots \int_a^b K(x,u_1) \ldots K(u_1,u_2) \ldots K(u_{r-1},y) \times K(y,v_1) \ldots K(v_1,v_2) \ldots K(v_{m-r-1},t)\, dv_{m-r-1} \ldots dv_2 dv_1 dy du_{r-1} \ldots du_2 du_1 \qquad (12)$$

[on changing the order of integration]

Note that the order of the multiple integral on R.H.S. of (8) is 1 + (r – 1) + (m – r – 1),that is, m – 1 . We have already proved that the order of the multiple integral on R.H.S. of (3.1.3) is also m – 1. Thus, multiple integrals involved in $K_m(x, t)$ and $\int_a^b K_r(x,y)K_{m-r}(y,t)dy$ are both of the same order, namely, $(m-1)^{th}$.

Now, changing the variables of integrations in (8) without changing the

$$\begin{array}{cccccccc} & u_1 u_2 & \ldots\ldots & u_{r-1} & y & v_1 v_2 & \ldots\ldots & v_{m-r-1} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & s_1 s_2 & \ldots\ldots & s_{r-1} s_r s_{r+1} s_{r+2} & & \ldots\ldots & s_{m-1} & \end{array}$$

limits of integration according to the following scheme

We obtain

$$\int_a^b K_r(x,y)K_{m-r}(y,t)dy$$

$$= \underbrace{\int_a^b \int_a^b \ldots \int_a^b}_{\substack{(m-1)th\ order \\ integral}} K(x,s_1)K(s_1,s_2)\ldots K(s_{r-1},s_r)K(s_{r-1},s_r)$$

$$\times K(s_r,s_{r+1})\ldots K_1(s_{m-1},t)ds_{m-1}\ldots ds_2 ds_1 \qquad (13)$$

From (7) and (13), we obtain

$$\mathrm{K_m(x,t)} = \int_a^b K_r(x,y)K_{m-r}(y,t)dy.$$

Check your Progress-1

1. Explain Fredholm Alternative Theorem.

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2. Define Iterated Kernels or Functions

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3. Provide statement of theorem for Resolvent Kernel

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3.6 LET US SUM UP

We also discussed Fredholm's alternative and an approximated method for solving Fredholm's integral equation of the second kind with separable kernel.

3.7 KEYWORDS

5. An **arbitrary function** simply means that it is a function that you **are** free to define in **any way** you want

6. **Multiple Integrals -** is a generalization of the usual integral in one dimension to functions of multiple variables in higher-dimensional spaces, which is an integral of a function over a two-dimensional region. The most common multiple integrals are double and triple integrals, involving two or three variables

3.8 QUESTIONS FOR REVIEW

1. Solve the equation by considering separately all exceptional cases.

$$y(x) = \lambda \int_0^{2\pi} e^{iw(x-t)}(t)dt,$$

2. Construct the resolvent kernels for the following kernels for specified a and b.

$$(i)\ K(x,t) = \sin x\ \cos t\,; a = 0, b = \frac{\pi}{2}\ .$$

$$(ii)\ K(x,t) = xe^t;\ a = -1, b = 1.$$

3. Find the resolvent kernel associated with the following kernels :

i. |x – t| in the interval (0, 1)

ii. cos (x+ t) in the interval (0, 2π)

3.9 SUGGESTED READINGS AND REFERENCES

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2. Linear Integral Equation: W.V. Lovitt (Dover).
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4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

3.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation – 3.2

2. Provide definition with the help of equation – 3.4

3.Provide statement – 3.5.1

UNIT-4 METHOD OF SUCCESSIVE APPROXIMATIONS

STRUCTURE

4.0 OBJECTIVES

Understand the Method of Successive Approximations for Fredholm Integral Equation of the Second Kind

Comprehend the Conditions of Convergence

Enumerate the Uniqueness of Series Solution

4.1 INTRODUCTION

By converting integral equation of the first kind to a linear equation of the second kind and the ordinary differential equation to integral equation we are going to solve the equation easily. The method of successive approximations (Neumann's series) is applied to solve linear and nonlinear Volterra integral equation of the second kind. Some examples are presented to illustrate methods.

4.2 METHOD OF SUCCESSIVE APPROXIMATIONS FOR FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

4.2.1 Iterative Scheme

Consider Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \qquad (4.0)$$

As a zero-order approximation to the required solution y(x), let us take

$$y_0(x) = f(x) \qquad (4.1)$$

Further, if $y_n(x)$ and $y_{n-1}(x)$ are the n^{th} order and $(n-1)^{th}$-order approximations respectively, then these are connected by

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t)y_{n-1}(t)dt \qquad (4.2)$$

We know that the iterated kernels (or iterated functions) $K_n(x, t)$, (n = 1, 2, 3, . . .) are defined by

$$K_1(x,t) = K(x,t) \qquad (4.3)$$

And

$$K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz,\ n = 2, 3, \dots \qquad (4.4)$$

Putting n = 1 in (4.2), the first-order approximation, $y_1(x)$ is given by

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) y_0(t) dt \tag{4.5}$$

But from (4.1)

$$y_0(t) = f(t) \tag{4.6}$$

Substituting the above value of $y_0(t)$ in (4.2), we get

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt \tag{4.7}$$

Putting n = 2 in (4.2), the second-order approximation $y_2(x)$ is given by

$$y_2(x) = f(x) + \lambda \int_a^{-} K(x,t) y_1(t) dt$$

$$y_1(z) = f(z) + \lambda \int_a^b K(z,t) f(t) dt \tag{4.9}$$

Replacing x by z in (4. 7), we get

Substituting the above value of $y_1(z)$ in (3.28), we get

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) \left[f(z) + \lambda \int_a^b K(z,t) f(t) dt \right] dz$$

or

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) f(z) dz + \lambda^2 \int_a^b K(x,z) [\int_a^b K(z,t) f(t) dt] \, dz$$

or

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) f(z) dz + \lambda^2 \int_a^b K(x,z) \left[\int_a^b K(z,t) f(t) dt \right] dz \tag{4.10}$$

or

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z)f(z)dz + \lambda^2 \int_a^b f(t) \left[\int_a^b K(x,z)K(z,t)\, dz \right] dt$$

[On changing the order of integration in third term on R.H.S of (4.10)] or

$$y_2(x) = f(x) + \lambda \int_a^b K_1(x,t)f(t)dt + \lambda^2 \int_a^b K_2(x,t)f(t)dt \qquad (4.11)$$

[using (4.3) and (4.4)]

$$y_2(x) = f(x) + \sum_{m=1}^{2} \lambda^m \int_a^b K_m(x,t)f(t)dt$$

Proceeding likewise, we easily obtain by Mathematical induction the nth approximate solution $y_n(x)$ of (4.10) as

$$y_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,t)f(t)dt \qquad (4.12)$$

Proceeding to the limit as $n \rightarrow \infty$, we obtain the so called Neumann series.

$$y(x) = \lim_{n\to\infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t)f(t)dt \qquad (4.13)$$

We now determine the resolvent kernel (or reciprocal kernal) R(x, t; λ) or Γ(x, t; λ) in terms of the iterated kernels $K_n(x, t)$.For this purpose, by changing the order of integration and summation in the so called Neumann series (4.13), we obtain

$$y(x) = f(x) + \lambda \int_a^b \left[\sum_{m=1}^{n} \lambda^{m-1} K_m(x,t) \right] f(t)dt \qquad (4.14)$$

Comparing (4.14) with

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(\mathrm{t}) dt \qquad (4.15)$$

Here

$$R(x,t;\lambda) = \sum_{m=1}^{n} \lambda^{m-1} K_m(x,\mathrm{t}). \qquad (4.16)$$

4.3 CONDITIONS OF CONVERGENCE

For the Condition of convergence of (4.13), consider the partial sum (4.12) and apply the Schwarz inequality to the general term of this series. This leads us to

$$\left| \int_a^b K_m(x,\mathrm{t}) f(\mathrm{t}) dt \right|^2 \leq \left(\int_a^b |K_m(x,\mathrm{t})|^2 dt \right) \int_a^b |f(t)|^2 dt \qquad (4.17)$$

Let

$$\mathrm{D} = \text{norm of } f(t) = \left[\int_a^b |f(t)|^2 dt \right]^{\frac{1}{2}} \qquad (4.18)$$

Further, let C_m^2 denote the upper bound of the integral $\int_a^b |K_m(x,t)|^2\, dt$ so that

$$\int_a^b |K_m(x,t)|^2 dt \leq C_m^2 \qquad (4.19)$$

Using (3.18) and (3.19), (3.20) reduces to

$$\left| \int_a^b K_m(x,t) f(t) dt \right|^2 \leq C_m^2 D^2 \qquad (4.20)$$

Now, applying the Schwarz inequality to relation

$$K_m(x,t) = \int_a^b K_{m-1}(x,z)K(x,t)dt, \tag{4.21}$$

We get

$$|K_m(x,t)|^2 \leq \left(\int_a^b |K_{m-1}(x,z)|^2 dz\right) \times \left(\int_a^b |K(z,t)|^2 dz\right),$$

which when integrated with respect to t, gives

$$\int_a^b |K_m(x,t)|^2 dt \leq B^2 C_{m-1}^2, \tag{4.22}$$

where

$$B^2 = \int_a^b \int_a^b |K(x,t)|^2 \, dx \, dt. \tag{4.23}$$

The inequality (4.22) gives rise to the recurrence relation

$$C_m^2 \leq B^{2m-2} C_1^2. \tag{4.24}$$

Using (4.21) and (4.24), we get

$$\left|\int_a^b K_m(x,t)f(t)dt\right|^2 \leq C_1^2 D^2 B^{2m-2}. \tag{4.25}$$

showing that the general term of the partial sum (4.12) has a magnitude less than the quantity $DC_1|\lambda|^m B^{m-1}$. Hence the infinite series (4.13) converges faster than the geometric series with common ratio $|\lambda|$ B. It follows that, if the condition

$$|\lambda| B < 1 \quad \text{or} \quad |\lambda| < 1 \Big/ \left[\int_a^b \int_a^b |K(x,t)|^2 \, dx \, dt\right]^{\frac{1}{2}} \tag{4.26}$$

is satisfied, then the series (4.13) will be uniformly convergent.

Notes

Check your Progress-1

1. Discuss Iterative Scheme

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2. Explain Conditions of Convergence

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4.4 UNIQUENESS OF SERIES SOLUTION

Uniqueness of Solution for a given λ: If possible, let (4.10) possess two solutions $y_1(x)$ and $y_2(x)$. Then we have

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) y_1(t) dt \qquad (4.27)$$

and

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) y_2(t) dt \qquad (4.28)$$

Let

$$y_1(x) - y_2(x) = \phi(x). \qquad (4.29)$$

Subtracting (4.28) from (4.27), we have

$$y_1(x) - y_2(x) = \lambda \int_a^b K(x,t)[y_1(t) - y_2(t)] dt$$

or

$$\phi(x) = \lambda \int_a^b K(x,t)\, \phi(t) dt, \text{using } (4.29) \qquad (4.30)$$

which is homogeneous integral equation. Applying the Schwarz inequality to (4.30), we have

$$|\phi(x)|^2 \le |\lambda|^2 \left(\int_a^b |K(x,t)|^2 dt\right) \times \left(\int_a^b |\phi(x)|^2 dx\right)$$

Integrating with respect to *x*, gives

$$\int_a^b |\phi(x)|^2 dx \le |\lambda|^2 \left(\int_a^b \int_a^b |K(x,t)|^2 dx\, dt\right) \times \left(\int_a^b |\phi(x)|^2 dx\right)$$

or

$$\int_a^b |\phi(x)|^2 dx \le |\lambda|^2 B^2 \int_a^b |\phi(x)|^2 dx, \qquad \text{by}(4.23)$$

or

$$1 - |\lambda|^2 B^2 \int_a^b |\phi(x)|^2 dx \le 0, \qquad (4.31)$$

giving $\emptyset\ (x)$= 0, using (4.24) or $y_1(x) - y_2(x) = 0$ or $y_1(x) = y_2(x)$ showing that (4.10) has a unique solution.

From the uniqueness of the solution of (3.20), we now proceed to show that the resolvent kernel R (x, t; λ) is also unique. If possible, let equation (4.10) have, with $\lambda = \lambda_0$, two resolvent kernels R_1 (x, t; λ) and R_2 (x, t; λ). In view of the uniqueness of the solution (4.10), an arbitrary function f(x) satisfies the identity

$$f(x) + \lambda_0 \int_a^b R_1(x,t;\lambda_0) f(t)dt \equiv f(x) + \lambda_0 \int_a^b R_2(x,t;\lambda) f(t)dt. \qquad (4.32)$$

Setting $F(x, t; \lambda_0) = R_1(x, t; \lambda_0) - R_2(x, t; \lambda_0)$ (4.32) reduces to

$$\int_a^b F(x,t;\lambda_0)\ f(t)dt \equiv 0 \qquad (4.33)$$

for an arbitrary function f (t). Let us choose f (t) = $\overline{F\ (x,t;\ \lambda_0)}$ with fixed x. Here $\overline{F\ (x,t;\ \lambda_0)}$ denotes the complex conjugate of $F(x, t; \lambda_0)$. Then (4.33) reduces to

$$\int_a^b |R(x,t;\lambda_0)|^2\ f(t)dt \equiv 0 \Longrightarrow\ F(x,t;\lambda_0) = 0 \Longrightarrow R_1(x,t;\lambda_0) - R_2(x,t;\lambda_0$$

$$\Longrightarrow R_1(x,t;\lambda_0) = R_2(x,t;\lambda_0)$$

showing that the resolvent kernel in unique. The above analysis can be summed up in the following basic theorem.

4.4.1 Theorem To each L_2 -kernel K (x, t) there corresponds a unique resolvent kernel R (x, t; λ) which is an analytic function of λ, regular at least inside the circle| λ | < B – 1, and represented by the power series

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t).$$

Furthermore, if f(x) is also an L_2 - function, then the unique L_2 - solution of the FredholmEquation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$$

valid in the circle $\lambda < B^{-1}$ is given by the formula

4.5 DIFFERENT TYPE OF EXAMPLES

Type 1. Determination of the resolvent kernel or reciprocal kernel R (z, t; λ) or Γ (z, t; λ) If K_n (x, t) be iterated kernels then R (z, t : λ) = Γ(z, t ; λ) = $\sum_{m=0}^{\infty} \lambda^{m-1}$

Example 1. Determine the resolvent kernels for the Fredholm integral equation having kernels:

(i) $K(x,t) = e^{x+t}; a = 0, b = 1.$

(ii) $K(x,t) = (1+x)(1-t);\ a = -1, b = 1.$

Sol. (i) Iterated kernels $K_m(x, t)$ are given by

$$K_1(x,t) = K(x,t) \qquad (4.34)$$

And

$$K_m(x,t) = \int_0^1 K(x,z)K_{m-1}(z,t)\ dz \quad (4.35)$$

From (4.34)

$$K_1(x,t) = K(x,t) = e^{x+t} \qquad (4.36)$$

$$K_2(x,t) = \int_0^1 K(x,z)K_1(z,t)dz = \int_0^1 e^{x+z}e^{x+t}\,dz, \text{using } (4.36)$$

Putting n = 2 in (4.36), we have

Putting n = 3 in (4.36), we have

$$= e^{x+t}\int_0^1 e^{2z}\,dz = e^{x+t}\left[\frac{1}{2}e^{2z}\right]_0^1 = e^{x+t}\left(\frac{e^2-1}{2}\right) \qquad (4.37)$$

$$K_3(x,t) = \int_0^1 K(x,z)K_1(z,t)\,dz = \int_0^1 e^{x+z}e^{z+t}\left(\frac{e^2-1}{2}\right)dz$$

$$= e^{x+t}\left(\frac{e^2-1}{2}\right)\int_0^1 e^{z+t}\left(\frac{e^2-1}{2}\right)^2, \text{as before} \qquad (4.38)$$

and so on. Observing (4.36), (4.37) and (4.36), we may write

$$K_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}, m = 1,2,3,\ldots. \qquad (4.39)$$

Now, the required resolvent kernel is given by

$$(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t) = \sum_{m=1}^{\infty}\lambda^{m-1}e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}$$

$$= e^{x+t}\sum_{m=1}^{\infty}\left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} \qquad (4.40)$$

But

$$\sum_{m=1}^{\infty}\left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = 1 + \left\{\frac{\lambda(e^2-1)}{2}\right\} + \left\{\frac{\lambda(e^2-1)}{2}\right\}^2 + \cdots$$

which is an infinite geometric series with common ratio $\{\lambda(e^2 - 1)\}/2$

$$\therefore \sum_{m=1}^{\infty} \left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = \frac{1}{1-\{\lambda(e^2-1)\}/2} = \frac{1}{2-\lambda(e^2-1)}, \qquad (4.41)$$

Provided

$$\left|\frac{\lambda(e^2-1)}{2}\right| < 1 \quad \text{or} \quad |\lambda| < \frac{2}{e^2-1}$$

Using (4.40) and (4.41), (4.42) reduces to

$$R\,(x,t;\lambda) = \frac{2e^{x+t}}{2-\lambda(e^2-1)} \quad (4.42) \qquad \text{Provided} \qquad |\lambda| < \frac{2}{e^2-1}$$

Part (ii) Iterated Kernels $K_m(x, t)$ are given by

$$K_1(x, t) = K\,(x, t) \qquad (4.43)$$

And

$$K_m(x,t) = \int_{-1}^{1} K\,(x,z) K_{m-1}(z,t)\,dz \qquad (4.44)$$

From (4.43)

$$K_1(x,t) = K\,(x,t) = (1+x)(1-t) \qquad (4.45)$$

Putting n = 2 in (3.64), we have

$$K_2(x,t) = \int_{-1}^{1} K\,(x,z`) K_1(z,t)\,dz$$

$$= \int_0^1 (1+x)(1-z)(1+z)(1-t)\,dz, \text{by}(4.45)$$

$$= (1+x)(1-t)\int_0^1 (1-z^2)\,dz = (1+x)(1-t)\left[z - \frac{1}{3}e^3\right]_{-1}^{1}$$

$$K_2(x,t) = (2/3) \times (1+x)(1-t) \tag{4.46}$$

Next, putting n = 3 in (4.44), we have

$$K_3(x, \quad t) = \int_{-1}^{1} K(x.z)K_2(z,t)\,dz = \int_{-1}^{1} (1+x)(1-z).\frac{2}{3}(1+z)\,(1-t)\,dz$$

$$= \frac{2}{3}(1+x)(1-t)\int_{-1}^{1}(1-z^2)dz = \left(\frac{2}{3}\right)^2 (1+x)(1-t), \text{ as before} \tag{4.47}$$

and so on. Observing (4.44), (4.45) and (4.44), we may write

$$K_m(z,t) = \left(\frac{2}{3}\right)^{m-1}(1+x)(1-t) \tag{4.48}$$

Now, the required resolvent kernel is given by

$$R(x,t;\,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1}K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1}\left(\frac{2}{3}\right)^{m-1}(1+x)(1-t), \text{by}(4.48)$$

$$= (1+x)(1-t)\sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1} \tag{4.49}$$

But $\quad \sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1} = 1 + \frac{2\lambda}{3} + \left(\frac{2\lambda}{3}\right)^2 + \left(\frac{2\lambda}{3}\right)^3 + \cdots$

which is an infinite geometric series with common ratio (2 /3)

$$\sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1} = \frac{1}{1-(2\lambda/3)} = \frac{1}{3-2\lambda}, \tag{4.50}$$

Provided $\quad |2\lambda/3| < 1 \quad$ or $\quad |\lambda| < 3/2 \quad$ (4.51)

Using (4.50) and (4.51), (4.52) reduces to

$$R(x,t;\,\lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}, \quad \text{provided} \quad |\lambda| < 3/2 \tag{4.52}$$

Type 2: Solution of Fredholm integral equation with help of the resolvent kernel. Working Rule: Let

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)dt \qquad (4.53)$$

Be given Fredholm integral equation. Let K_m (x, t) be mth iterated kernel and let R (x, t, λ) be the resolvent kernel of (4.53). Then we have

$$R(x,t,\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \qquad (4.54)$$

Suppose the sum of infinite series (4.54) exists and so R (x, t; λ)can be obtained in the closed form. Then, the required solution of (4.53) is given by

$$y(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \qquad (4.55)$$

Type 3: Solution of Fredholm integral equation when the resolvent kernel cannot be obtained in closed form i.e., the sum of infinite series occurring in the formula of the resolvent kernel cannot be determined. In such integral equation, we use the method of successive approximations to find solutions upto third order.

Working Rule : Let the given Fredholm integral equation of the second kind be

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) dt \qquad (4.56)$$

As zero-order approximation, we take

$$y_0(x) = f(x) \qquad (4.57)$$

If n^{th}- order approximation be y_n (x), then

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t) y_{n-1}(t) dt \qquad (4.58)$$

Remark. Sometimes the zero-order approximation is mentioned in the problem. In that case,we modify equation (4.57) according to data of the problem.

Check your Progress-2

3. Explain Uniqueness of Series Solution

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4. State the theorem of Uniqueness of Series Solution

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4.6 LET US SUM UP

In this chapter we have discussed Method of Successive Approximations for Fredholm Integral Equation of the second kind. In which iterative scheme is discussed. As we know that convergence and uniqueness is an important phenomenon so conditions of convergence and uniqueness of series solution is also discussed

4.7 KEYWORDS

1. **Complex Conjugate** - each of two complex numbers having their real parts identical and their imaginary parts of equal magnitude but opposite sign.

2. **Kernel of an integral equation:** The function K(x, y) in the above equations is called the kernel of the equation
3. **Geometric Series:** a geometric series is a series with a constant ratio between successive terms**.**

4.8 QUESTIONS FOR REVIEW

1. Solve

$$y(x) = x + \int_0^{1/2} y(t)\, dt$$

2. Solve the following integral equations by the method of successive approximations:

(i) $y(x) = \frac{3x}{6} + \frac{1}{2}\int_0^1 xt\, y(t)\, dt$

(ii) $y(x) = x + \lambda \int_0^1 xt\, y(t)\, dt$

3. Using iterative method, solve

$$y(x) = f(x) + \lambda \int_0^1 e^{x-t}\, y(t) dt$$

4.9 SUGGESTED READINGS AND REFERENCES

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2. Linear Integral Equation: W.V. Lovitt (Dover).
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4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

4.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation – 4.2.1

2. Provide explanation – 4.3

3.Provide explanation – 4.4

4. Provide statement – 4.4.1

UNIT-5 FREDHOLM THEOREMS

STRUCTURE

5.0 OBJECTIVES

Understand the Fredholm equation

Comprehend Fredholm's First, Second and Third Theorem

5.1 INTRODUCTION

In mathematics, **Fredholm theory** is a theory of integral equations. In the narrowest sense, Fredholm theory concerns itself with the solution of the Fredholm integral equation. In a broader sense, the abstract structure of Fredholm's theory is given in terms of the spectral theory of Fredholm operators and Fredholm kernels on Hilbert space. The theory is named in honour of Erik Ivar Fredholm.

5.2 WHAT IS FREDHOLM EQUATION?

Fredholm gave the solution of equation

$$g(s) = f(s) + \lambda \int K(s,t)g(t)dt \qquad (5.1)$$

in general form for all values of the parameter λ. His results are contained in three theorems which bear his name.

The method used by Fredholm consists in viewing the integral equation (5.1) as the limiting case of a system of linear algebraic equations. This theory applies to two-or higher-dimensional integrals, although we shall confine our discussion to only one dimensional integrals in the interval (a, b) . Let us divide the interval (a, b)into equal parts

$s_1 = t_1 = a$, $s_2 = t_2 = a+h$, …, $s_n = t_n = a + (n – 1) h$

$$\int K(s,t)g(t)dt \simeq h \sum_{j=1}^{n} K(s,s_j)g(s_j). \qquad (5.2)$$

where $h = \frac{b-a}{n}$. Thereby, we have the approximate formula

Equation (5.1) then takes the form

$$g(s) \simeq f(s) + \lambda h \sum_{j=1}^{n} K(s,s_j)g(s_j), \qquad (5.3)$$

which must hold for all values of s in the interval (a, b)

In particular, this equation is satisfied at the n points of division s_i , i = 1,2, … , n.This leads to the system of equations

$$g(s_i) = f(s_i) + \lambda h \sum_{j=1}^{n} K(s_i,s_j)g(s_j), \ \ i = 1, \dots, n. \qquad (5.4)$$

$$f(s_i) = f_i, \qquad g(s_i) = g_i, \ K(s_i, s_j) = K_{ij}, \tag{5.5}$$

equation (5.4) yields an approximation for the integral equation (4.1) in terms of the system of linear equations

$$g_i - \lambda h \sum_{j=1}^{n} K_{ij} g_j = f_i\,, \qquad i = 1, \dots, n. \tag{5.6}$$

inunknown quantities $g_1, g_2, \ldots, g_n$. The values of Ã=obtained by solving this algebraic system are approximate solutions of the integral equation (1) at the points $s_1, s_2, \ldots, s_n$ We can plot these solutions g_i as ordinates and by interpolation draw a curve g(s)which we may expect to be an approximation to the actual solution. With the help of this algebraic system, we can also determine approximations for the eigen values of the kernel.

The resolvent determinant of the algebraic system (5.6) is

$$D_n(\lambda) = \begin{vmatrix} 1 - \lambda h K_{11} & -\lambda h K_{12} & \dots & -\lambda h K_{1n} \\ -\lambda h K_{21} & 1 - \lambda h K_{22} & \dots & -\lambda h K_{2n} \\ & \vdots & & \\ -\lambda h K_{n1} & -\lambda h K_{n2} & \dots & 1 - \lambda h K_{nn} \end{vmatrix} \tag{5.7}$$

The approximate eigenvalues are obtained by setting this determinant equal to zero. We illustrate it by the following example

Example. Consider

$$g(s) - \lambda \int_0^{\pi} \sin(s + t) g(t) dt = 0.$$

By taking n = 3, we have $h = \frac{\pi}{3}$ and therefore

$$s_1 = t_1 = 0, s_2 = t_2 = \frac{\pi}{3}, s_3 = t_3 = \frac{2\pi}{3},$$

and the values of K_{ij} are readily calculated to give

$$(K_{ij}) = \begin{vmatrix} 0 & 0.866 & 0.866 \\ 0.866 & 0.866 & 0 \\ 0.866 & 0 & -0.866 \end{vmatrix}.$$

The homogeneous system corresponding to (4.6) will have a nontrivial solution if the determinant

$$D_n(\lambda) = \begin{vmatrix} 1 & -0.907\lambda & -0.907\lambda \\ -0.907\lambda & (1-0.907\lambda) & 0 \\ -0.907\lambda & 0 & 1+0.907\lambda \end{vmatrix} = 0,$$

or when $1 - 3(0.0907)^2\lambda^2 = 0$ The roots of this equation are $\lambda = \pm 0.6365$. This gives a rather close agreement with the exact values, which are $\pm\frac{\sqrt{2}}{\pi} = \pm 0.6366$. In general, the practical applications of this method are limited because one has to take a rather large n to get a reasonable approximation.

5.3 FREDHOLM'S FIRST THEOREM

The solutions $g_1, g_2, \ldots, g_n$ of the system of equations (5.6) are obtained as ratios of certain determinants, with the determinant $D_n(\lambda)$ given by (5.7) as the denominator provided it does not vanish. Let us expand the determinant (5.7) in powers of the quantity $(-\lambda h)$. The constant term is obviously equal to unity. The term containing $(-\lambda h)$ in the first power is the sum of all the determinants containing only one column $-\lambda h K_{\mu v}$, $\mu = 1, 2, \ldots, n$. Taking the contribution from all the columns $v = 1, \ldots, n$, we find that the total contribution is $-\lambda h \sum_{v=1}^{n} K_{vv}$.

The factor containing the factor $(-\lambda h)$ to the second power is the sum of all the determinants containing two columns with that factor. This results in the determinants of the form

$$(-\lambda h)^2 \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

where (p, q) is an arbitrary pair of integers taken from the sequence 1, … , n, with p < q. In the same way, it follows that the term containing the factor $(-\lambda h)^3$ is the sum of the determinants of the form

$$(-\lambda h)^3 \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix}$$

Where (p, q, r)is an arbitrary triplet of integers selected from the sequence 1, … , , with p < q < r.

The remaining terms are obtained in a similar manner. Therefore, we conclude that the required expansion of $D_n(\lambda)$is

$$D_n(\lambda) = 1 - \lambda h \sum_{v=1}^{n} K_{vv} + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

$$+ \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix} + \cdots$$

$$+ \frac{(-\lambda h)^n}{n!} \sum_{p_1,p_2,\dots,p_n=1}^{n} \begin{vmatrix} K_{p_1p_1} & K_{p_1p_2} & \dots & K_{p_1p_n} \\ K_{p_2p_1} & K_{p_2p_2} & \dots & K_{p_2p_n} \\ & \vdots & & \\ K_{p_np_1} & K_{p_np_2} & \dots & K_{p_np_n} \end{vmatrix}, \qquad (5.8)$$

where we now stipulate that the sums are taken over all permutations of pairs (p, q), triplets (p,q,r), etc. This convention explains the reason for dividing each term of the above series by the corresponding number of permutations.

The analysis is simplified by introducing the following symbol for the determinant formed by the values of the kernel at all points (si, tj)the so-

$$\begin{vmatrix} K(s_1,t_1) & K(s_1,t_2) & \cdots & K(s_1,t_n) \\ K(s_2,t_1) & K(s_2,t_2) & \cdots & K(s_2,t_n) \\ & \vdots & & \\ K(s_n,t_1) & K(s_n,t_2) & \cdots & K(s_n,t_n) \end{vmatrix} = K\begin{pmatrix} s_1, s_2, \cdots, s_n \\ t_1, t_2, \cdots, t_n \end{pmatrix}, \qquad (5.9)$$

called Fredholm determinant. We observe that, if any pair of arguments in the upper or lower sequence is transposed, the value of the determinant changes sign because the transposition of two arguments in the upper sequence corresponds to the transposition of two rows of the determinant and the transposition of two arguments in the lower sequence corresponds to the transposition of two columns.

In this notation, the series (5.8) takes the form

$$D_n(\lambda) = 1 - \lambda h \sum_{p=1}^{n} K(s_p, s_p) + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} K\begin{pmatrix} s_p s_q \\ s_p s_q \end{pmatrix} \qquad (5.10)$$

$$+ \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} K\begin{pmatrix} s_p, s_q, s_r \\ s_p, s_q, s_r \end{pmatrix} + \cdots.$$

If we now let n tend to infinity, then *h* will tend to zero, and each term of the sum (5.10) tends to some single, double, triple integral, etc. There results Fredholm's first series:

$$D(\lambda) = 1 - \lambda \int K(s,s)ds + \frac{\lambda^2}{2!} \int K\begin{pmatrix} s_1 s_2 \\ s_1 s_2 \end{pmatrix} ds_1 ds_2$$

$$-\frac{\lambda^3}{3!} \iiint K\begin{pmatrix} s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 \end{pmatrix} ds_1 ds_2 ds_3 + \cdots. \qquad (5.11)$$

Hilbert gave a rigorous proof of the fact that the sequence Dn(λ) → D (λ) in the limit, while the convergence of the series (5.11) for all values of was proved by Fredholm on the basis that the kernel K(s,t) is a boundedand integrable function. Thus, D (λ) is an entire function of the complex variable λ.

We are now ready to solve the Fredholm equation (5.1) and express the solutions in the form of a quotient of two power series in the parameter λ, where the Fredholm function D (λ) is to be the divisor. We seek solutions of the form

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)dt, \qquad (5.12)$$

and expect the resolvent kernel Γ(s, t; λ) to be the quotient

$$\Gamma(s,t;\lambda) = \mathrm{D}(s,t;\lambda)/D(\lambda), \tag{5.13}$$

where D(s, t ; λ)still to be determined, is the sum of certain functional series. Now, the resolvent Γ(s, t; λ) itself satisfies a Fredholm integral equation of the second kind :

$$\Gamma(s,t;\lambda) = K(s,t) + \lambda \int K(s,x)\,\Gamma(x,t;\lambda)dx. \tag{5.14}$$

$$\mathrm{D}(s,t;\lambda) = K(s,t)D(\lambda) + \lambda \int K(s,x)\,\mathrm{D}(x,t;\lambda)dx. \tag{5.15}$$

From (5.13) and (5.14), it follows that

The form of the series (5.11) for D(λ)suggests that we seek the solution of equation (5.15) in the form of a power series in the parameter λ

$$\mathrm{D}(s,t;\lambda) = C_0(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} C_p(s,t). \tag{5.16}$$

For this purpose, write the numerical series (5.11) as

$$\mathrm{D}(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} c_p. \tag{5.17}$$

where

$$\mathrm{c_p} = \underbrace{\iint \dots \int}_{p-times} K\begin{pmatrix} s_1, s_2, \cdots, s_p \\ s_1, s_2, \cdots, s_p \end{pmatrix} ds_1 \dots ds_p. \tag{5.18}$$

The next step is to substitute the series for D(s, t ; λ)and ä(‹)from (5.16) and (5.17) in (5.15) and compare the coefficients of equal powers of λ. The following relations result:

$$C_0(s,t) = K(s,t), \tag{5.19}$$

$$C_p(s,t) = c_p K(s,t) - p\int K(s,x)C_{p-1}(x,t)dx. \tag{5.20}$$

$$C_p(s,t) = \iint \dots \int K\begin{pmatrix} s, x_1, x_2, \cdots, x_p \\ t, x_1, x_2, \cdots, x_p \end{pmatrix} dx_1 \dots dx_p. \tag{5.21}$$

In fact, for p = 1, the relation (5.20) becomes

$$C_1(s,t) = c_1 K(s,t) - \int K(s,x)C_0(x,t)dx$$

$$= K(s,t)\int K(x,x)dx - \int K(s,x)K(x,t)dx$$

$$= \int K\begin{pmatrix} s & x \\ t & x \end{pmatrix} dx, \tag{5.22}$$

where we have used (5.18) and (5.19). To prove that (5.21) holds for general "we expand the determinant under the integral sign in the relation:

$$K\begin{pmatrix} s, x_1, x_2, \cdots, x_p \\ t, x_1, x_2, \cdots, x_p \end{pmatrix} = \begin{vmatrix} K(s,t) & K(s,x_1) & \cdots & K(s,x_p) \\ K(x_1,t) & K(x_1,x_1) & \cdots & K(x_1,x_p) \\ & \vdots & & \\ K(x_p,t) & K(x_p,x_1) & \cdots & K(x_p,x_p) \end{vmatrix},$$

with respect to the elements of the given row, transposing in turn the first column one place to the right, integrating both sides, and using the definition of c_p in (5.18); the required result then follows by induction.

$$D(s,t;\lambda) = K(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \cdots \int K\begin{pmatrix} s, x_1, x_2, \cdots, x_p \\ t, x_1, x_2, \cdots, x_p \end{pmatrix} dx_1 dx_2 \dots dx_p. \tag{5.23}$$

From (5.16), (5.19), and (5.21) we derive Fredholm's second series:

Notes

This series also converges for all values of the parameter .It is interesting to observe the similarity between the series (5.11) and (5.23). Having found both terms of the quotient (5.13), we have established the existence of a solution to the integral equation (5.1) for a bounded and integrable kernel K (s, t) provided, of course, that D(λ) = 0. Since both terms of this quotient are entire functions of the parameter it follows that the resolvent kernel Γ(s, t; λ) is a meromorphic function of , i.e., an analytic function whose singularities may only be the poles, which in the present case are zeros of the divisor D(λ) .Next, we prove that the solution in the form obtained by Fredholm is unique and is given by

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t) dt. \tag{5.24}$$

In this connection, we first observe that the integral equation (4.14) satisfied by Γ(s, t; λ) is valid for all values of for which D(λ) ≠ 0.Indeed, (5.14) is known to hold for $|\lambda| < B^{-1}$, and since both sides of this equation are now proved to be meromorphic, the above contention follows. To prove the uniqueness of the solution, let us suppose that g(s) is a solution of the equation (5.1) in the case D(λ) ≠ 0. Multiply both

$$\int \Gamma(s,x;\lambda) g(x) dx = \int \Gamma(s,x;\lambda) f(x) dx + \lambda \int \left[\int \Gamma(s,x;\lambda) K(x,t) dx \right] g(t) dt \tag{5.25}$$

sides of (5.1) by Γ(s, t; λ) integrate, and get

Substituting from (5.14) into left side of (5.25), this becomes which, when joined by (5.1), yields

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t) dt. \tag{5.27}$$

but this form is unique. In particular, the solution of the homogeneous equation

$$\int K(s,t)\, g(t) dt = \int \Gamma(s,x;\lambda) f(x) dx, \tag{5.26}$$

$$g(s) = \lambda \int \mathrm{K}(\mathrm{s},\mathrm{t})\mathrm{g}(t)dt. \qquad (5.28)$$

is identically zero. The above analysis leads to the following theorem.

Fredholm's First Theorem. The inhomogeneous Fredholm equation

$$g(s) = f(s) + \lambda \int K(s,t)g(t)dt, \qquad (5.29)$$

where the functions f(s) and g(t) are integrable, has a unique solution

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda)f(t)dt, \qquad (5.30)$$

where the resolvent kernel Γ(s, t; λ)

$$\Gamma(s,t;\lambda) = \mathrm{D}(s,t;\lambda)/D(\lambda), \qquad (5.31)$$

with D (λ) ≠ 0, is a meromorphic function of the complex variable λ being the ratio of two entire functions defined by the series

$$\mathrm{D}(s,t;\lambda)$$

$$= K(s,t) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \cdots \int \mathrm{K}\begin{pmatrix} s, x_1, x_2, \cdots, x_p \\ t, x_1, x_2, \cdots, x_p \end{pmatrix} dx_1 dx_2 \dots dx_p, \qquad (5.32)$$

and

$$\mathrm{D}(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int \cdots \int K\begin{pmatrix} x_1, x_2, \cdots, x_p \\ x_1, x_2, \cdots, x_p \end{pmatrix} dx_1 dx_2 \dots dx_p, \qquad (5.33)$$

both of which converge for all values of λ. In particular, the solution of the homogeneous equation

Notes

$$g(s) = \lambda \int K(s,t)g(t)dt \tag{5.34}$$

is identically zero.

EXAMPLES

Example 1. Evaluate the resolvent for the integral equation

$$g(s) = f(s) + \lambda \int_0^1 (s+t)g(t)dt.$$

The solution to this example is obtained by writing

$$\Gamma(s,t;\lambda) = \frac{\left[\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} C_p(s,t)\right]}{\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} c_p}$$

Where C_p and c_p are defined by the relations (5.18) and (5.20)

$$c_p = \int C_{p-1}(s,s)ds,$$

$$C_p = c_p K(s,t) - p\int_0^1 K(s,x)c_{p-1}(x,t)dx \tag{5.35}$$

$$c_1 = \int_0^1 2sds = 1,$$

Thus

$$c_1 = \int_0^1 2sds = 1,$$

$$C_1(s,t) = (s+t) - \int_0^1 (s+x)(x+t)dx = \frac{1}{2}(s+t) - st - \frac{1}{3},$$

$$c_2 = \int_0^1 \left(s - s^2 - \frac{1}{3}\right) ds = -\frac{1}{6},$$

$$C_2(s,t) = -\frac{1}{6}(s+t) - 2\int_0^1 (s+x)\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right] dx = 0$$

Since C_2 (x, t) vanishes, it follows from (5.35) that the subsequent coefficients C_k land c_k also vanish. Therefore,

$$\Gamma(s,t;\lambda) = \frac{(s+t) - \left[\frac{1}{2}(s+t) - st - \frac{1}{3}\right]\lambda}{1 - \lambda - \frac{\lambda^2}{12}}.$$

5.4 FREDHOLM'S SECOND THEOREM

Feedhole's first theorem does not hold when is a root of the equation ä() = 0. We know that, for a separable kernel, the homogeneous equation

$$g(s) = \lambda \int K(s,t)g(t)dt \qquad (5.36)$$

has nontrivial solutions. It might be expected that same holds when the kernel is an arbitrary integrable function and we shall then have a spectrum of eigen values and corresponding eigen functions. The second theorem of Fredholm is devoted to the study of this problem.

We first prove that every zero of D(λ)is a pole of the resolvent kernel (5.31); the order of this pole is at most equal to the order of the zero of ä(). In fact, differentiate the Fredholm's first series (5.33) and interchange the indices of the variables of integration to get

$$D'(\lambda) = -\int D(s,s;\lambda)ds. \qquad (5.37)$$

From this relation, it follows that, if qis a zero of order ,of D(λ), then it is a zero of order , − 1 of D'(λ) and consequently λ_0 may be zero of order at most ,k − 1 of the entire function D(s, t; λ)Thus, λ_0 is the pole of the quotient (5.31) of order at most ,. In particular, if λ_0 is a simple zero of D(λ), then $D(\lambda_0) = 0$, $D'(\lambda_0) \neq 0$, and λ_0 is a simple pole of the resolvent kernel. Moreover, it follows from (5.37) that D(s, t; λ) ≠ 0. For this particular case, we observe from equation (5.8) that, if D (λ) = 0 and D (s, t; λ)≠ 0, then D(s, t; λ), as a function of 0, is a solution of the

Notes

homogeneous equation (5.36). So is αD(s, t; λ)where α is an arbitrary constant. Let us now consider the general case when is a zero of an arbitrary multiplicity +, that is, when

$$D(\lambda_0) = 0, \dots, D^r(\lambda_0) = 0, D^m(\lambda_0) \neq 0, \qquad (5.38)$$

where the superscript cstands for the differential of order c, c = 1, . . . , + − 1. For this case, the analysis is simplified if one defines a determinant known as the Fredholm minor:

$$D_n\left(\begin{matrix} s_1, & s_2, & \dots, & s_n \\ t_1, & t_2, & \dots, & t_n \end{matrix}\middle| \lambda\right) = K\left(\begin{matrix} s_1, & s_2, & \dots, & s_n \\ t_1, & t_2, & \dots, & t_n \end{matrix}\right) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!}$$

$$\times \int \dots \int K\left(\begin{matrix} s_1, \dots, s_n, x_1, \dots, x_p \\ t_1, \dots, t_n, x_1, \dots, x_p \end{matrix}\right) \, dx_1 \cdots dx_p, \qquad (5.39)$$

where{s_i} and {t_i}, i = 1,2, … ,n, are two sequences of arbitrary variables. Just as do the Fredholm series (5.32) and (5.33), the series (5.39) also converges for all values of and consequently is an entire function of λ. Furthermore, by differentiating the series (5.33) times and comparing it with the series (5.39), there follows the relation

$$\frac{d^n D(\lambda)}{d\lambda^n} = (-1)^n \int \cdots \int D_n\left(\begin{matrix} s_1, \cdots, s_n \\ s_1, \cdots, s_n \end{matrix}\middle| \lambda\right) ds_1 ds_2 \cdots ds_n. \qquad (4.40)$$

From this relation, we conclude that, if λ_0 is a zero of multiplicity m of the function D(λ), then the following holds for the Fredholm minor of order m for that value of λ_0:

$$D_m\left(\begin{matrix} s_1, s_2, \cdots, s_m \\ t_1, t_2, \cdots, t_m \end{matrix}\middle| \lambda_0\right) \neq 0.$$

Of course, there might exist minors of order lower than m which also do not identically vanish. Let us find the relation among the minors that corresponds to the resolvent formula (5.14). Expansion of the determinant under the integral sign in (5.39),

$$\begin{vmatrix} K(s_1,t_1)\ K(s_1,t_2)\cdots K(s_1,t_n)\ K(s_1,x_1)\ \cdots K(s_1,x_p) \\ K(s_2,t_1)\ K(s_2,t_2)\cdots K(s_2,t_n)\ K(s_2,x_1)\ \cdots K(s_2,x_p) \\ \vdots \\ K(s_n,t_1)\ K(s_n,t_2)\cdots K(s_n,t_n)\ K(s_n,x_1)\ \cdots K(s_n,x_p) \\ K(x_1,t_1)\ K(x_1,t_2)\cdots K(x_1,t_n)\ K(x_1,x_1)\ \cdots K(x_1,x_p) \\ \vdots \\ K(x_p,t_1)\ K(x_p,t_2)\cdots K(x_p,t_n)\ K(x_p,x_1)\ \cdots K(x_p,x_p) \end{vmatrix}. \qquad (5.41)$$

by elements of the first row and integrating "times with respect to x_1, x_2, …, x_p for $p \geq 1$, we have

$$\int \dots \int K\begin{pmatrix} s_1,\dots,s_n,x_1,\dots,x_p \\ t_1,\dots,t_n,x_1,\dots,x_p \end{pmatrix} dx_1 \dots dx_p = \sum_{h=1}^{n} (-1)^{h+1} K(s_1,t_h)$$

$$\times \int \dots \int K\begin{pmatrix} s_1,\dots,s_h,\cdots, & s_n,x_1,\dots,x_p \\ t_1,\dots,t_{h-1},t_{h+1},\cdots,t_n,x_1,\dots,x_p \end{pmatrix} dx_1 \dots dx_p = \sum_{h=1}^{p} (-1)^{h+n-1}$$

$$\times \int \dots \int K(s_1,x_h) K\begin{pmatrix} s_2,\dots, & s_n,x_1,x_2,\dots,x_h,\cdots,x_p \\ t_1,\dots,t_{n-1},t_n,x_1,\dots,x_{h-1},x_{h+1},x_p \end{pmatrix} dx_1 \dots dx_p$$

$$\times dx_1,\cdots,dp. \qquad (5.42)$$

Note that the symbols for the determinant on the right side of (4.42) do not contain the variables s_1 in the upper sequence and the variables t_h or x_h in the lower sequence. Furthermore, it follows by transposing the variable s_h in the upper sequence to the first place by means of h + n− 2 transpositions that all the components of the second sum on the right side are equal. Therefore, we can write (5.42) as

$$\int \dots \int K\begin{pmatrix} s_1,\dots,s_n,x_1,\dots,x_p \\ t_1,\dots,t_n,x_1,\dots,x_p \end{pmatrix} dx_1 \dots dx_p = \sum_{h=1}^{n} (-1)^{h+1} K(s_1,t_h)$$

$$\times \int \dots \int K\begin{pmatrix} s_2,\dots, & s_n,x_1,\dots,x_p \\ t_1,\dots,t_{h-1},t_{h+1},t_n,x_1,\dots,x_p \end{pmatrix} dx_1 \dots dx_p$$

$$-p \int K(s_1,x) \Big[\int \cdots \int K\begin{pmatrix} x,s_2,\dots,s_n,x_1,\dots,x_{p-1} \\ t_1,t_2,\dots,t_n,x_1,\dots,x_{p-1} \end{pmatrix}$$

$$\times dx_1 \cdots dx_{p-1}\Big] dx. \qquad (5.43)$$

Notes

where we have omitted the subscript *h* from x. Substituting (5.43) in (5.42), we find that Fredholm minor satisfies the integral equation

$$D_n\left(\begin{matrix} s_1,\cdots,s_n \\ t_1,\cdots,t_n \end{matrix}\middle|\lambda\right)=\sum_{h=1}^{n}(-1)^{h+1}K(s_1,t_h)D_{n-1}\left(\begin{matrix} s_2,\dots, & & s_n \\ t_1,\dots,t_{h-1},t_{h+1},t_n \end{matrix}\right)$$

$$+\lambda\int K(s_1,x)D_n\left(\begin{matrix} x,s_2,\cdots,s_n \\ t_1,t_2\cdots,t_n \end{matrix}\middle|\lambda\right)dx. \tag{5.44}$$

Expansion by the elements of any other row leads to a similar identity, with x placed at the corresponding place. If we expand the determinant (5.41) with respect to the first column and proceed as above, we get the integral equation

$$D_n\left(\begin{matrix} s_1,\cdots,s_n \\ t_1,\cdots,t_n \end{matrix}\middle|\lambda\right)=\sum_{h=1}^{n}(-1)^{h+1}K(s_h,t_1)D_{n-1}\left(\begin{matrix} s_1,\dots,s_{h-1},s_{h+1},\cdots,s_n \\ t_2,\dots \qquad\qquad ,t_n \end{matrix}\right)$$

$$+\lambda\int K(x,t_1)D_n\left(\begin{matrix} s_1,\cdots \quad ,s_n \\ x,t_2\cdots,t_n \end{matrix}\right)dx, \tag{5.45}$$

and a similar result would follow if we were to expand by any other column. The formulas (5.44) and (5.45) will play the role of the Fredholm series of the previous section.

Note that the relations (5.44) and (5.45) hold for all values of . With the help of (5.44), we can find the solution of the homogeneous equation (5.36) for the special case when $\lambda = \lambda_0$ is an eigenvalue. To this end, let us suppose that $\lambda = \lambda_0$ is a zero of multiplicity m of the function $D(\lambda)$. Then, as remarked earlier, the minor D_m does not identically vanish and even the minors $D_1, D_2, \dots D_{m-1}$ may not identically vanish. Let D_r be the first minor in the sequence $D_1, D_2, \dots D_{m-1}$ that does not vanish identically. The number r lies between 1 and m and is the index of the eigenvalue λ_0. Moreover, this means that $D_{r-1} = 0$. But then the integral equation (5.44) implies that

$$g_1(s)=D_r\left(\begin{matrix} s,s_2,\cdots,s_r \\ t_1,\cdots,t_r \end{matrix}\middle|\lambda_0\right) \tag{5.46}$$

is a solution of the homogeneous equation (5.36). Substituting s at different points of the upper sequence in the minor D_r, we obtain r nontrivial solutions $g_i(s)$ = (0), = 1, ⋯ , r, of the homogeneous equation. These solutions are usually written as

$$\Phi_i(s) = \frac{D_r\begin{pmatrix} s_1, \cdots, s_{i-1}, s, s_{i+1}, \cdots, s_r \\ t_1, \cdots \quad\quad\quad\quad , t_r \end{pmatrix}\lambda_0}{D_r\begin{pmatrix} s_1, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_r \\ t_1, \cdots \quad\quad\quad\quad , t_r \end{pmatrix}\lambda_0}, i = 1,2, \cdots, r. \quad (5.47)$$

Observe that we have already established that the denominator is not zero. The solutions Φ_i as given by (5.47) are linearly independent for the following reason. In the determinant (5.41) above, if we put two of the arguments 0=equal, this amounts to putting two rows equal, and consequently the determinant vanishes. Thus, in (5.47), we see that $\Phi_k(s_i) = 0$ for $i \neq k$,, whereas $\Phi_k(s_k) = 1$. Now, if there exists a relation $\sum_k C_k \Phi_k \equiv 0$ we may put $s = s_i$= and it follows that $C_i \equiv 0$; and this proves the linear independence of these solutions. This system of solutions Φ_i is called the fundamental system of the eigen functions of $_q$ and any linear combination of these functions gives a solution of (5.36). Conversely, we can show that any solution of equation (5.36) must be a linear combination of $\Phi_1(s), \Phi_2(s), \cdots, \Phi_v(s)$. We need to define a kernel H(s, t; λ)which corresponds to the resolvent kernel Γ(s, t; λ)of the previous section

$$H(s,t;\lambda) = \frac{D_{r+1}\begin{pmatrix} s, s_1, \cdots, s_r \\ t, t_1, \cdots, t_r \end{pmatrix}\lambda_0}{D_r\begin{pmatrix} s_1, \cdots, s_r \\ t_1, \cdots, t_r \end{pmatrix}\lambda_0} . \quad (5.48)$$

In (4.45), take n to be equal to r, and add extra arguments s and t to obtain

$$D_{r+1}\begin{pmatrix} s, s_1, \cdots, s_r \\ t, t_1, \cdots, t_r \end{pmatrix}\lambda_0 = K(s,t) D_r \begin{pmatrix} s_1, \cdots, s_r \\ t_1, \cdots, t_r \end{pmatrix}\lambda_0$$

$$+ \sum_{h=1}^{r} (-1)^h K(s_h, y) D_r \begin{pmatrix} s, s_1, \cdots, s_{h-1}, s_{h+1}, \cdots, s_r \\ t_1, t_2, \cdots \quad\quad\quad\quad , t_r \end{pmatrix}\lambda_0$$

$$+\lambda_0 \int K(x,t)\, D_{r+1}\begin{pmatrix} s, s_1, \cdots, s_r \\ t, t_1, \cdots, t_r \end{pmatrix}\lambda_0 \quad dx. \quad (5.49)$$

Notes

In every minor D_r in the above equation, we transpose the variable 0from the first place to the place between the variables s_{h-1} and s_{h+1} and divide both sides by the constant

$$D_r\begin{pmatrix} s_1, \cdots, s_r \\ t_1, \cdots, t_r \end{pmatrix} \lambda_0 \Big) \neq 0,$$

To obtain

$$= -\sum_{h=1}^{r} K(s_h, t)\Phi_h(s). \qquad (5.50)$$

If g(s) is any solution to (5.36), we multiply (5.50) by g(t) and integrate with respect to t,

$$\int g(t)H(s,t;\lambda)dt - \frac{g(s)}{\lambda_0} - \int g(x)\Gamma(s,x;\lambda)dx$$

$$= -\sum_{h=1}^{r} \frac{g(s_h)}{\lambda_0}\Phi_h(s), \qquad (5.51)$$

where we have used (5.36) in all terms but the first; we have also taken

$\lambda_0 \int K(s_h, t)g(t)dt = g(s_h)$.

Cancelling the equal terms, we have

$$g(s) = \sum_{h=1}^{r} g(s_h)\Phi_h(s) \qquad (5.52)$$

This proves our assertion. Thus we have established the following result.

Fredholm's Second Theorem.

If λ_0 is a zero of multiplicity m of the function $D(\lambda)$ then the homogeneous equation

$$g(s) = \lambda_0 \int K(s,t)g(t)dt \qquad (5.53)$$

possesses at least one, and at most m, linearly independent solutions

$$g_i(s) = D_r\left(\begin{matrix} s_1, \cdots, s_{i-1}, s, s_{i+1}, \cdots, s_r \\ t_1, \cdots \qquad\qquad\qquad, t_r \end{matrix}\middle| \lambda_0\right), i = 1, \cdots, r; 1 \le r \le m \quad (5.54)$$

not identically zero. Any other solution of this equation is a linear combination of these solutions.

5.5 FREDHOLM'S THIRD THEOREM

Fredholm's Third Theorem. For an inhomogeneous equation

$$g(s) = f(s) + \lambda_0 \int K(s,t)g(t)dt \qquad (5.55)$$

to possess a solution in the case $D(\lambda_0) = 0$, it is necessary and sufficient that the given function f(s) be orthogonal to all the eigenfunctions $\Psi_i(s)$, i = 1,2, ⋯ , v, of the transposed homogeneous equation corresponding to the eigenvalue λ_0. The general solution has the form

$$g(s) = f(s) + \lambda_0 \int \frac{\left[D_{r+1}\left(\begin{matrix} s, s_1, \cdots, s_r \\ t, t_1, \cdots, t_r \end{matrix}\middle| \lambda_0\right)\right]}{D_r\left(\begin{matrix} s_1, \cdots, s_r \\ t_1, \cdots, t_r \end{matrix}\middle| \lambda_0\right)} \times f(t)dt + \sum_{h=1}^{r} C_h \Phi_h(s).$$

Check your Progress-1

1. Discuss Fredholm's First Theorem

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2. Explain Fredholm's Second Theorem

3. State Fredholm's Third Theorem

5.6 LET US SUM UP

We have studied three different theorems viz., Fredholm's First Theorem Fredholm's SecondTheorem Fredholm's Third Theorem.

5.7 KEYWORDS

4. meromorphic function - **A** meromorphic **function** is a single-valued **function** that is analytic **in** all but **possibly** a discrete subset of its domain, **and at** those singularities it must **go** to **infinity** like **a** polynomial (i.e.**,** these exceptional points must be poles and not **essential** singularities).
5. complex variable - **a variable** that can take **on the value of** a **complex number.**
6. Parameter - a numerical or other measurable factor forming one of a set that defines a system or sets the conditions of its operation.

5.8 QUESTIONS FOR REVIEW

1. Solve the integral equation using Fredholm's First Theorem

$$g(s) = 1 + \lambda \int_0^{\pi} [\sin(s+t)] g(t) dt,$$

2. Solve the integral equation using Fredholm's First Theorem

$$g(s) = s + \lambda \int_0^1 \left[st + (st)^{\frac{1}{2}}\right] g(t)dt.$$

5.9 SUGGESTED READINGS AND REFERENCES

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5. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
6. D. Powers, Boundary Value Problems Academic Press, 1979.

5.10 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation – 5.3

2. Provide explanation – 5.4

3. Provide statement – 5.5

UNIT-6 GREEN FUNCTION I

STRUCTURE

6.0 OBJECTIVES

Understand the Green Function concept

Comprehend the method of variation of parameters

Comprehend Initial and Boundary Value Green's Functions

6.1 INTRODUCTION

Green's functions are named after the British mathematician George Green, who first developed the concept in the 1830s. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead.

In mathematics, a **Green's function** is response to the elementary impulse of an inhomogeneous linear differential operator, which is defined on a domain with specified initial or boundary conditions.

This means that if L is the linear differential operator, then

- the Green's function G is the solution of the equation LG = δ, where δ is Dirac's delta function;
- the solution of the initial-value problem Ly = f is the convolution (G * f), where G is the Green's function.

6.2 CONCEPT OF GREEN FUNCTION

Our goal is to solve the nonhomogeneous differential equation

$$L[u] = f,$$

where L is a differential operator. The solution is formally given by $u = L^{-1}[f]$.

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$u = \int G(x, \xi) f(\xi)\, d\xi.$$

The function G(x, ξ) is referred to as the kernel of the integral operator and is called the Green's function.

6.3 THE METHOD OF VARIATION OF PARAMETERS

We are interested in solving non-homogeneous second order linear differential equations of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x). \qquad (6.1)$$

The general solution of this non-homogeneous second order linear differential equation is found as a sum of the general solution of the homogeneous equation,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad (6.2)$$

and a particular solution of the nonhomogeneous equation.

However, a more methodical method, which is first seen in a first course in differential equations, is the Method of Variation of Parameters. We will review this method in this section and extend it to the solution of boundary value problems.

While it is sufficient to derive the method for the general differential equation above, we will instead consider solving equations that are in SturmLiouville, or self-adjoint, form. Therefore, we will apply the Method of Variation of Parameters to the equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \qquad (6.3)$$

Note that f(x) in this equation is not the same function as in the general equation posed at the beginning of this section.

We begin by assuming that we have determined two linearly independent solutions of the homogeneous equation. The general solution is then given by

$$y(x) = c_1y_1(x) + c_2y_2(x). \qquad (6.4)$$

In order to determine a particular solution of the nonhomogeneous equation, we vary the parameters c1 and c2 in the solution of the homogeneous problem by making them functions of the independent variable. Thus, we seek a particular solution of the nonhomogeneous equation in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \qquad (6.5)$$

In order for this to be a solution, we need to show that it satisfies the differential equation. We first compute the derivatives of yp(x). The first derivative is

$y_p'(x) = c_1(x)\, y_1'(x) + c2(x)\, y_2'(x) + c_1'(x) y_1(x) + c_2'(x) y_2(x).$

Without loss of generality, we will set the sum of the last two terms to zero. Then, we have

$$c_1'(x) y_1(x) + c_2'(x) y_2(x) = 0. \tag{6.6}$$

Now, we take the second derivative of the remaining terms to obtain

$y_p''(x) = c_1(x)\, y_1''(x) + c2(x)\, y_2''(x) + c_1'(x)\, y_1'(x) + c_2'(x)\, y_2'(x).$

Expanding the derivative term in Equation (6.3),

$$p(x)\, y_p''(x) + p'(x)\, y_p'(x) + q(x) y_p(x) = f(x),$$

and inserting the expressions for yp, $y_p'(x)$, and $y_p''(x)$, we have

$$\begin{aligned} f(x) = {} & c_1(x)\left[p(x)y_1''(x) + p'(x)y_1'(x) + q(x)y_1(x)\right] \\ & + c_2(x)\left[p(x)y_2''(x) + p'(x)y_2'(x) + q(x)y_2(x)\right] \\ & + p(x)\left[c_1'(x)y_1'(x) + c_2'(x)y_2'(x)\right]. \end{aligned} \tag{6.7}$$

Since y1(x) and y2(x) are both solutions of the homogeneous equation. The first two bracketed expressions vanish. Dividing by p(x), we have that

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{p(x)}. \tag{6.8}$$

Our goal is to determine c1(x) and c2(x). In this analysis, we have found that the derivatives of these functions satisfy a linear system of equations (in the ci's)

Linear System for Variation of Parameters

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0. \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \end{aligned} \tag{6.9}$$

This system is easily solved to give

$$c_1'(x) = -\frac{f(x)y_2(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]}$$
$$c_2'(x) = \frac{f(x)y_1(x)}{p(x)\left[y_1(x)y_2'(x) - y_1'(x)y_2(x)\right]}. \qquad (6.10)$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem. Recall that

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Furthermore, we can show that the denominator, p(x)W(x), is constant. Differentiating this expression and using the homogeneous form of the differential equation proves this assertion

$$\begin{aligned} \frac{d}{dx}(p(x)W(x)) &= \frac{d}{dx}\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right] \\ &= y_1(x)\frac{d}{dx}(p(x)y_2'(x)) + p(x)y_2'(x)y_1'(x) \\ &\quad - y_2(x)\frac{d}{dx}(p(x)y_1'(x)) - p(x)y_1'(x)y_2'(x) \\ &= -y_1(x)q(x)y_2(x) + y_2(x)q(x)y_1(x) = 0. \end{aligned} \qquad (6.11)$$

$$c_1(x) = -\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi$$
$$c_2(x) = \int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi, \qquad (6.12)$$

where x0 and x1 are arbitrary constants to be determined later.

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi. \qquad (6.13)$$

Therefore, the particular solution of (6.3) can be written as

As a further note, we usually do not rewrite our initial value problems in self-adjoint form. Recall that for an equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \qquad (6.14)$$

we obtained the self-adjoint form by multiplying the equation by

$$\frac{1}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}\,dx} = \frac{1}{a_2(x)}p(x).$$

This gives the standard form

(p(x)y′(x))′ + q(x)y(x) = f(x),

$$f(x) = \frac{1}{a_2(x)}p(x)g(x).$$

With this in mind, Equation (6.13) becomes

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{g(\xi)y_1(\xi)}{a_2(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{g(\xi)y_2(\xi)}{a(\xi)W(\xi)}\,d\xi. \qquad (6.15)$$

Example. Consider the nonhomogeneous differential equation

y″ − y′ − 6y = 20e−2x.

We seek a particular solution to this equation. First, we note two linearly independent solutions of this equation are

$y_1(x) = e^{3x}$, $y_2(x) = e^{-2x}$.

So, the particular solution takes the form
$y_p(x) = c_1(x)e^{3x} + c_2(x)e^{-2x}$.

We just need to determine the c_i's. Since this problem is not in self-adjoint form, we will use

$$\frac{f(x)}{p(x)} = \frac{g(x)}{a_2(x)} = 20e^{-2x}$$

as seen above. Then the linear system we have to solve is

$$\begin{aligned} c_1'(x)e^{3x} + c_2'(x)e^{-2x} &= 0. \\ 3c_1'(x)e^{3x} - 2c_2'(x)e^{-2x} &= 20e^{-2x}. \end{aligned} \qquad (6.16)$$

Multiplying the first equation by 2 and adding the equations yields

$$5c_1'(x)e^{3x} = 20e^{-2x},$$

$$c_1'(x) = 4e^{-5x}.$$

Inserting this back into the first equation in the system, we have

$$4e^{-2x} + c_2'(x)e^{-2x} = 0,$$

$$c_2'(x) = -4.$$

These equations are easily integrated to give

$$c_1(x) = -\frac{4}{5}e^{-5x}, \quad c_2(x) = -4x.$$

Therefore, the particular solution has been found as

$$\begin{aligned} y_p(x) &= c_1(x)e^{3x} + c_2(x)e^{-2x} \\ &= -\frac{4}{5}e^{-5x}e^{3x} - 4xe^{-2x} \\ &= -\frac{4}{5}e^{-2x} - 4xe^{-2x}. \end{aligned} \tag{6.17}$$

Noting that the first term can be absorbed into the solution of the homogeneous problem. So, the particular solution can simply be written as

$$y_p(x) = -4xe^{-2x}.$$

This is the answer you would have found had you used the Modified Method of Undetermined Coefficients.

Check your Progress-1

1. Define Green Function

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2. Give generalized equation for Linear System for Variation of Parameters

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6.4 INITIAL AND BOUNDARY VALUE GREEN'S FUNCTIONS

We begin with the particular solution (6.13) of our nonhomogeneous differential equation (6.3). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_2(x) \int_{x_1}^{x} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(x) \int_{x_0}^{x} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi. \tag{6.18}$$

As seen in the last section, an appropriate choice of x_0 and x_1 could be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(x)+ c_2y_2(x)$. However, setting up the solution in this form will allow us to use x_0 and x_1 to determine particular solutions which satisfies certain homogeneous conditions. We will now consider initial value and boundary value problems. Each type of problem will lead to a solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_a^b G(x, \xi) f(\xi)\, d\xi, \tag{6.19}$$

where the function $G(x, \xi)$ will be identified as the Green's function and the integration limits will be found on the integral. Having identified the Green's function, we will look at other methods in the last section for determining the Green's function.

6.4.1 Initial Value Green's Function

We begin by considering the solution of the initial value problem

Notes

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x).$$
$$y(0) = y_0, \quad y'(0) = v_0. \tag{6.20}$$

Of course, we could have studied the original form of our differential equation without writing it in self-adjoint form. However, this form is useful when studying boundary value problems. We will return to this point later.

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$\frac{d}{dx}\left(p(x)\frac{dy_h(x)}{dx}\right) + q(x)y_h(x) = 0.$$
$$y_h(0) = y_0, \quad y_h'(0) = v_0. \tag{6.21}$$

We then assume that the particular solution satisfies the problem

$$\frac{d}{dx}\left(p(x)\frac{dy_p(x)}{dx}\right) + q(x)y_p(x) = f(x).$$
$$y_p(0) = 0, \quad y_p'(0) = 0. \tag{6.22}$$

Since the differential equation is linear, then we know that y(x) = yh(x)+ yp(x) is a solution of the nonhomogeneous equation. However, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$
$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on solving for the particular solution that satisfies homogeneous initial conditions. Recall Equation (6.13) from the last section,

$$y_p(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{6.23}$$

We now seek values for x_0 and x_1 which satisfies the homogeneous initial conditions, $y_p(0) = 0$ and $y'_p(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0)\int_{x_1}^{0}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(0)\int_{x_0}^{0}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \qquad (6.24)$$

Here, $y_1(x)$ and $y_2(x)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then we have

$$y_p(0) = y_2(0)\int_{x_1}^{0}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \qquad (6.25)$$

We can force $y_p(0) = 0$ if we set $x_1 = 0$.

Now, we consider $y'_p(0) = 0$. First we differentiate the solution and find that

$$y'_p(x) = y'_2(x)\int_{0}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y'_1(x)\int_{x_0}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi, \qquad (6.26)$$

since the contributions from differentiating the integrals will cancel.

Evaluating this result at $x = 0$, we have

$$y'_p(0) = -y'_1(0)\int_{x_0}^{0}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \qquad (6.27)$$

Assuming that $y'_1(0) \neq 0$, we can set $x0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(x)\int_{0}^{x}\frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{0}^{x}\frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \\ &= \int_{0}^{x}\left[\frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}\right]f(\xi)\,d\xi. \end{aligned} \qquad (6.28)$$

This result is in the correct form and we can identify the temporal,or initial value, Green's function. So, the particular solution is given as

$$y_p(x) = \int_0^x G(x,\xi) f(\xi)\, d\xi, \qquad (6.29)$$

where the initial value Green's function is defined as

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}.$$

We summarize

Solution of Initial Value Problem (6.20)

The solution of the initial value problem (8.21) takes the form

$$y(x) = y_h(x) + \int_0^x G(x,\xi) f(\xi)\, d\xi, \qquad (6.30)$$

where

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{p(\xi)W\xi)}$$

and the solution of the homogeneous problem satisfies the initial conditions,

$$y_h(0) = y_0, \quad y_h'(0) = v_0.$$

Example: Solve the forced oscillator problem

$x'' + x = 2 \cos t$, $x(0) = 4$, $x'(0) = 0$.

Solution: We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \qquad x_h(0) = 4, \qquad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4 \cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, to the homogeneous differential equation satisfying $y_1(0) = 0$ and $y_2'(0) = 0$. So, we pick $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$W(t) = y_1(t)\, y_2'(t) - y_1'(t) y_2(t) = -\sin^2 t - \cos^2 t = -1.$

Since $p(t) = 1$ in this problem, we have

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{p(\tau)W\tau)} \\ &= \sin t \cos\tau - \sin\tau \cos t \\ &= \sin(t-\tau). \end{aligned} \qquad (6.31)$$

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form.

We can now determine the particular solution of the non-homogeneous differential equation. We have

$$x_p(t) = \int_0^t G(t,\tau)f(\tau)\,d\tau$$
$$= \int_0^t (\sin t\cos\tau - \sin\tau\cos t)\,(2\cos\tau)\;d\tau$$

$$= 2\sin t\int_0^t \cos^2\tau d\tau - 2\cos t\int_0^t \sin\tau\cos\tau d\tau$$
$$= 2\sin t\left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \qquad (6.32)$$
$$= t\sin t.$$

Therefore, the particular solution is $x(t) = 4\cos t + t\sin t$. As noted, we usually are not given the differential equation in self-adjoint form. Generally, it takes the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x). \qquad (6.33)$$

The driving term becomes

$$f(x) = \frac{1}{a_2(x)}p(x)g(x).$$

Inserting this into the Green's function form of the particular solution, we obtain the following:

Solution Using the Green's Function

The solution of the initial value problem,

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

takes the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \int_0^t G(x,\xi)g(\xi)\,d\xi, \qquad (6.34)$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \frac{y_1(\xi)y_2(x) - y_1(x)y_2(\xi)}{a_2(\xi)W(\xi)} \qquad (6.35)$$

and $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0.$$

6.4.2 Boundary Value Green's Function

We now turn to boundary value problems. We will focus on the problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,$$
$$y(a) = 0, \quad y(b) = 0. \tag{6.36}$$

However, the general theory works for other forms of homogeneous boundary conditions. Once again, we seek x_0 and x_1 in the form

$$y(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi$$

so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Here we absorb $y_h(x)$ into the integrals with an appropriate choice of lower limits on the integrals. We first pick solutions of the homogeneous differential equation such that $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. So, we have

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(a)\int_{x_0}^{a} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{6.37}$$

This expression is zero if $x_1 = a$.

At $x = b$ we find that

$$\begin{aligned} y(b) &= y_2(b)\int_{x_1}^{b} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= -y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \end{aligned} \tag{6.38}$$

This vanishes for $x_0 = b$.

So, we have found that

$$y(x) = y_2(x)\int_{a}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\,d\xi - y_1(x)\int_{b}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \tag{6.39}$$

We are seeking a Green's function so that the solution can be written as one integral. We can move the functions of x under the integral. Also, since a < x < b, we can flip the limits in the second integral. This gives

$$y(x) = \int_a^x \frac{f(\xi)y_1(\xi)y_2(x)}{p(\xi)W(\xi)}\,d\xi + \int_x^b \frac{f(\xi)y_1(x)y_2(\xi)}{p(\xi)W(\xi)}\,d\xi. \qquad (6.40)$$

This result can be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem takes the form

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi, \qquad (6.41)$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \frac{y_1(\xi)y_2(x)}{pW}, & a \le \xi \le x \\ \frac{y_1(x)y_2(\xi)}{pW} & x \le \xi \le b \end{cases}. \qquad (6.42)$$

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at x = a and x = b. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{pW} = 0,$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{pW} = 0.$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x) = \begin{cases} \frac{y_1(x)y_2(\xi)}{pW}, & a \le x \le \xi \\ \frac{y_1(\xi)y_2(x)}{pW} & \xi \le x \le b \end{cases}. \qquad (6.43)$$

But a careful look at the original form shows that

$$G(x, \xi) = G(\xi, x).$$

Notes

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example: Solve the boundary value problem y″ = x2, y(0) = 0 = y(1) using the boundary value Green's function.

We first solve the homogeneous equation, y″ = 0. After two integrations, we have y(x) = Ax + B, for A and B constants to be determined.

We need one solution satisfying $y_1(0) = 0$ Thus, $0 = y_1(0) = B$. So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So, $0 = y_2(1) = A + B$. This can be solved for B = −A. Again, A is arbitrary and we will choose A = −1. Thus, $y_2(x) = 1 - x$.

For this problem p(x) = 1. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = x(-1) - 1(1 - x) = -1.$$

Note that p(x)W(x) is a constant, as it should be. Now we construct the Green's function. We have

$$G(x,\xi) = \begin{cases} -\xi(1-x),\ 0 \le \xi \le x \\ -x(1-\xi),\ x \le \xi \le 1 \end{cases}. \tag{6.44}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: G(0, ξ) = 0, from the lower branch, and G(1, ξ) = 0, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution:

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^1 G(x,\xi)\xi^2\, d\xi \\ &= -\int_0^x \xi(1-x)\xi^2\, d\xi - \int_x^1 x(1-\xi)\xi^2\, d\xi \\ &= -(1-x)\int_0^x \xi^3\, d\xi - x\int_x^1 (\xi^2 - \xi^3)\, d\xi \\ &= -(1-x)\left[\frac{\xi^4}{4}\right]_0^x - x\left[\frac{\xi^3}{3} - \frac{\xi^4}{4}\right]_x^1 \\ &= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3 - 3x^4) \\ &= \frac{1}{12}(x^4 - x). \end{aligned} \tag{6.45}$$

Check your Progress-2

1. State Solution of Initial Value Green's Function

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2. Explain Boundary Value Green's Function

6.5 LET US SUM UP

We understood the concept of Green function. We also comprehended important solution of Green Function using Initial Value and Boundary conditions.

6.6 KEYWORDS

1. In functional analysis, a linear operator A on a Hilbert space is called **self-adjoint** if it is equal to its own **adjoint** A^* and that the domain of A is the same as that of A^*.
2. **Initial condition**. : any of a set of starting-point **values** belonging to or imposed upon the variables in an equation that has one or more arbitrary constants.
3. **Arbitrary** means "undetermined; not assigned a specific value.

6.7 QUESTIONS FOR REVIEW

1. Use the Method of Variation of Parameters to determine the general solution for the following problems.

a. $y'' + y = \tan x$.

b. $y'' - 4y' + 4y = 6xe^{2x}$.

Notes

2. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}$, $y(0) = 0$, $y'(0) = 6$.

b. $y'' + y = 2 \sin 3x$, $y(0) = 5$, $y'(0) = 0$.

3. Solve the boundary value problem using the Green's function.

$$y'' = \sin x, \qquad y'(0) = 0, \qquad y(\pi) = 0.$$

6.8 SUGGESTED READINGS AND REFERENCES

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4. The Use of Integral Transform, I.n. Sneddon, Tata-McGrawHill, 1974
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6. D. Powers, Boundary Value Problems Academic Press, 1979.

6.9 ANSWERS TO CHECK YOUR PROGRESS

1. Provide explanation – 6.2

2. Provide equations– 6.3 [Refer equation – 6.9]

3. Provide explanation– 6.4.1[Refer equation – 6.30]

4. Provide explanation– 6.4.2

UNIT-7 GREEN FUNCTION II

STRUCTURE

7.0 OBJECTIVES

Understand different properties of Green's Function and its application

Understand the concept The Dirac Delta Function

Enumerate Green's Function Differential Equation

7.1 INTRODUCTION

We have noted some properties of Green's functions in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems.

7.2 PROPERTIES OF GREEN'S FUNCTIONS

Properties of the Green's Function

1. **Differential Equation:**
$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$
For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.
2. **Boundary Conditions:**
For $x = a$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x,\xi)$ will satisfy. A similar statement can be made for $x = b$.
3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi,x)$
We had shown this in the last section.

4. **Continuity of G at** $x = \xi$: $G(\xi^+,\xi) = G(\xi^-,\xi)$
Here we have defined

$$G(\xi^+,x) = \lim_{x\downarrow\xi} G(x,\xi), \quad x > \xi,$$

$$G(\xi^-,x) = \lim_{x\uparrow\xi} G(x,\xi), \quad x < \xi.$$

Setting $x = \xi$ in both branches, we have

$$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x = \xi$:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\begin{aligned}\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} &= -\frac{1}{pW}y_1(\xi)y_2'(\xi) + \frac{1}{pW}y_1'(\xi)y_2(\xi)\\ &= -\frac{y_1'(\xi)y_2(\xi) - y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi))}\\ &= \frac{1}{p(\xi)}. \qquad (7.1)\end{aligned}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function.

Example. Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \qquad 0 < x < 1,$$

$$y(0) = 0 = y(1),$$

with $\omega \neq 0$.

I. Find solutions to the homogeneous equation.

A general solution to the homogeneous equation is given as

$$y_h(x) = c_1 \sin \omega x + c_2 \cos\omega x.$$

Thus, for $x \neq \xi$,

$$G(x, \xi) = c1(\xi) \sin \omega x + c2(\xi) \cos\omega x.$$

II. Boundary Conditions.

First, we have $G(0, \xi) = 0$ for $0 \leq x \leq \xi$. So,

$$G(0, \xi) = c_2(\xi) \cos \omega x = 0.$$

So,

$$G(x, \xi) = c_1(\xi) \sin \omega x,\ 0 \leq x \leq \xi.$$

Second, we have $G(1, \xi) = 0$ for $\xi \leq x \leq 1$. So,

$$G(1, \xi) = c_1(\xi) \sin \omega + c_2(\xi) \cos\omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi) \tan \omega.$$

This gives

$$G(x, \xi) = c1(\xi) \sin \omega x - c1(\xi) \tan \omega \cos\omega x.$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x,\xi) &= \frac{c_1(\xi)}{\cos\omega}[\sin\omega x \cos\omega - \sin\omega \cos\omega x] \\ &= -\frac{c_1(\xi)}{\cos\omega} \sin\omega(1-x). \qquad (7.2) \end{aligned}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x, \xi) = d1(\xi) \sin \omega(1 - x),\ \xi \leq x \leq 1.$$

Notes

We note that we could have started with y2(x) = sinω(1−x) as one of our linearly independent solutions of the homogeneous problem in anticipation that y2(x) satisfies the second boundary condition.

III. Symmetry or Reciprocity

We now impose that G(x, ξ) = G(ξ, x). To this point we have that

$$G(x,\xi) = \begin{cases} c_1(\xi)\sin\omega x, & 0 \le x \le \xi \\ d_1(\xi)\sin\omega(1-x), & \xi \le x \le 1 \end{cases}.$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi)$ = C sinω(1 − ξ) and $d_1(\xi)$ = C sin ωξ. Then,

$$G(x,\xi) = \begin{cases} C\sin\omega(1-\xi)\sin\omega x, \; 0 \le x \le \xi \\ C\sin\omega(1-x)\sin\omega\xi, \; \xi \le x \le 1 \end{cases}.$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where C = 1/pW .

IV. Continuity of G(x, ξ)

We note that we already have continuity by virtue of the symmetry imposed in the last step.

V. Jump Discontinuity in $\frac{\partial}{\partial x}$G(x, ξ).

We still need to determine C. We can do this using the jump discontinuity of the derivative

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}.$$

For our problem p(x) = 1. So, inserting our Green's function, we have

$$
\begin{aligned}
1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\
&= \frac{\partial}{\partial x}\left[C\sin\omega(1-x)\sin\omega\xi\right]_{x=\xi} - \frac{\partial}{\partial x}\left[C\sin\omega(1-\xi)\sin\omega x\right]_{x=\xi} \\
&= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\xi \\
&= -\omega C\sin\omega(\xi+1-\xi) \\
&= -\omega C\sin\omega.
\end{aligned} \tag{7.3}
$$

Therefore,

$$
C = -\frac{1}{\omega\sin\omega}.
$$

Finally, we have our Green's function:

$$
G(x,\xi) = \begin{cases} -\frac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, & 0 \le x \le \xi \\ -\frac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, & \xi \le x \le 1 \end{cases}. \tag{7.4}
$$

It is instructive to compare this result to the Variation of Parameters result. We have the functions $y_1(x) = \sin \omega x$ and $y_2(x) = \sin \omega(1 - x)$ as the solutions of the homogeneous equation satisfying y1(0) = 0 and y2(1) = 0. We need to compute pW:

$$
\begin{aligned}
p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\
&= -\omega\sin\omega x\cos\omega(1-x) - \omega\cos\omega x\sin\omega(1-x) \\
&= -\omega\sin\omega
\end{aligned} \tag{7.5}
$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

7.3 THE DIRAC DELTA FUNCTION

We will develop a more general theory of Green's functions for ordinary differential equations which encompasses some of the listed properties. The Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, \quad x \neq \xi. \tag{7.6}$$

When x = ξ, we saw that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x) = \begin{cases} 1, \; x > 0 \\ 0, \; x < 0 \end{cases}.$$

In the case of the step function, the derivative is zero everywhere except at the jump. At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this by introducing the Dirac delta function,

$$\delta(x) = \frac{d}{dx}H(x).$$

We will then show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{7.7}$$

The Dirac delta function, δ(x), is one example of what is known as a generalized function, or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Before defining the Dirac delta function and introducing some of its properties, we will look at some representations that lead to the definition. We will consider the limits of two sequences of functions.

First we define the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n} \\ \frac{n}{2}, & |x| < \frac{1}{n} \end{cases}$$

This is a sequence of functions as shown in Figure 7.1. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one since each box has height n/2 and width n/2.

Thus, the limiting function is zero at most points but has area one.

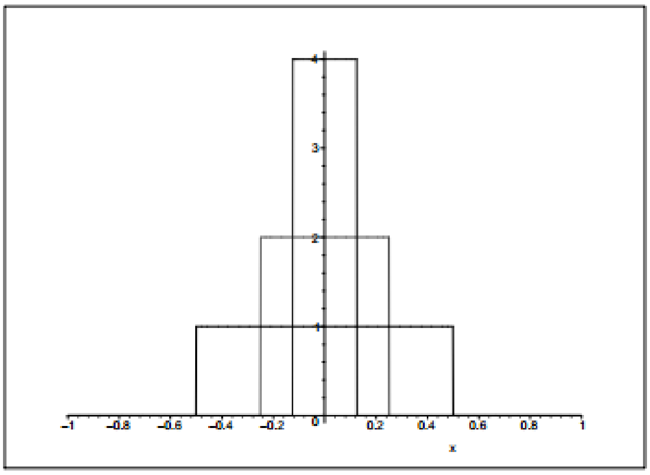

Fig. 7.1 . A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a generalized function. It is called the Dirac delta function, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Another example is the sequence defined by

$$D_n(x) = \frac{2\sin nx}{x}. \tag{7.8}$$

We can graph this function. We first rewrite this function as

$$D_n(x) = 2n\frac{\sin nx}{nx}.$$

Now it is easy to see that as $x \to 0$, $D_n(x) \to 2n$. For large x, The function tends to zero. A plot of this function is in Figure 7.2. For large n the peak grows and the values of $D_n(x)$ for $x \neq 0$ tend to zero as show in Figure 7.3.

We note that in the limit $n \to \infty$, $D_n(x) = 0$ for x 6= 0 and it is infinite at $x = 0$. However, using complex analysis one can show that the area is

$$\int_{-\infty}^{\infty} D_n(x)\,dx = 2\pi.$$

Thus, the area is constant for each n.

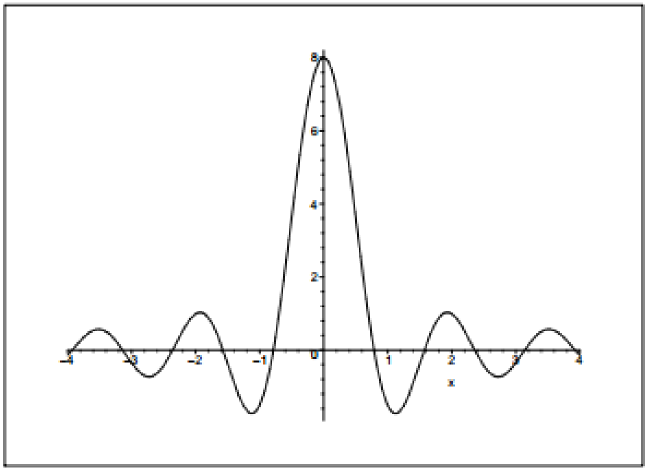

Fig. 7.2 A plot of the function $D_n(x)$ for $n = 4$.

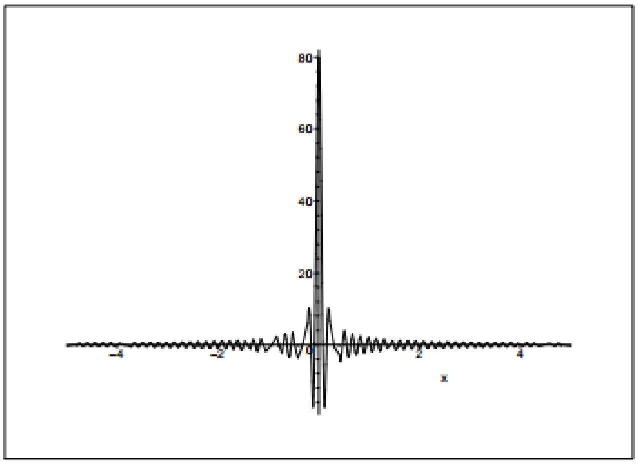

Fig. 7.3 A plot of the function $D_n(x)$ for $n = 40$.

There are two main properties that define a Dirac delta function. First one has that the area under the delta function is one

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)\,dx = 1, \quad 0 \in [a,b]$$

$$\int_a^b \delta(x)\,dx = 0, \quad 0 \notin [a,b].$$

Another common property is what is sometimes called the sifting property. Namely, integrating the product of a function and the delta function "sifts" out a specific value of the function. It is given by

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at x = a. Therefore, the integrand is zero everywhere and the only contribution from f(x) will be from x = a. So, we can replace f(x) with f(a) under the integral. Since f(a) is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\,dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a).$$

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1}\delta(x). \qquad (7.9)$$

As usual, this only has meaning under an integral sign. So, we place δ(ax) inside an integral and make a substitution y = ax:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(ax)\,dx &= \lim_{L\to\infty}\int_{-L}^{L} \delta(ax)\,dx \\ &= \lim_{L\to\infty}\frac{1}{a}\int_{-aL}^{aL} \delta(y)\,dy. \qquad (7.10)\end{aligned}$$

If a > 0 then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy.$$

However, if a < 0 then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a}\int_{\infty}^{-\infty} \delta(y)\, dy = -\frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\, dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor 1/a to 1/|a|. Thus,

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(y)\, dy. \qquad (7.11)$$

Example: Evaluate

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\, dx.$$

Solution : This is not a simple δ(x−a). So, we need to find the zeros of f(x) = 3x−2. There is only one, x = 2 /3. Also, |f ′(x)| = 3. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\, dx = \frac{1}{3}\int_{-\infty}^{\infty} \delta(y)\left(\frac{y+2}{3}\right)^2 dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for x_j, j = 1,2, . . ., n, (i.e.; when one has n simple zeros), then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|}\delta(x - x_j).$$

7.4 GREEN'S FUNCTION DIFFERENTIAL EQUATION

As noted, the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi) \qquad (7.12)$$

and satisfies homogeneous conditions. We have used the Green's function to solve the non-homogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \tag{7.13}$$

These equations can be written in the more compact forms

$$\begin{aligned} \mathcal{L}[y] &= f(x) \\ \mathcal{L}[G] &= \delta(x-\xi). \end{aligned} \tag{7.14}$$

Multiplying the first equation by G(x, ξ), the second equation by y(x), and then subtracting, we have

$$G\mathcal{L}[y] - y\mathcal{L}[G] = f(x)G(x,\xi) - \delta(x-\xi)y(x).$$

Now, integrate both sides from x = a to x = b. The left side becomes

$$\int_a^b [f(x)G(x,\xi) - \delta(x-\xi)y(x)]\, dx = \int_a^b f(x)G(x,\xi)\, dx - y(\xi)$$

and, using Green's Identity, the right side is

$$\int_a^b (G\mathcal{L}[y] - y\mathcal{L}[G])\, dx = \left[p(x)\left(G(x,\xi)y'(x) - y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}$$

Combining these results and rearranging, we obtain

$$y(\xi) = \int_a^b f(x)G(x,\xi)\, dx - \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \tag{7.15}$$

Next, one uses the boundary conditions in the problem in order to determine which conditions the Green's function needs to satisfy. For example, if we have the boundary condition y(a) = 0 and y(b) = 0, then the boundary terms yield

$$y(\xi) = \int_a^b f(x)G(x,\xi)\,dx - \left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi) - G(b,\xi)y'(b)\right)\right]$$
$$+ \left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi) - G(a,\xi)y'(a)\right)\right]$$
$$= \int_a^b f(x)G(x,\xi)\,dx + p(b)G(b,\xi)y'(b) - p(a)G(a,\xi)y'(a). \qquad (7.16)$$

The right hand side will only vanish if G(x, ξ) also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi) = \int_a^b f(x)G(x,\xi)\,dx.$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi)G(\xi,x)\,d\xi.$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, G(x, ξ) is symmetric in its arguments. So, we can simply switch the arguments getting the desired result. We can now see that the theory works for other boundary conditions. If we had y′(a) = 0, then the y(a) $\frac{\partial G}{\partial x}$(a, ξ) term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}$(a, ξ) = 0.

We can even adapt this theory to non- homogeneous boundary conditions.

We first rewrite Equation (7.15)

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi - \left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b}. \qquad (7.17)$$

Let's consider the boundary conditions y(a) = α and y′(b) = beta. We also assume that G(x, ξ) satisfies homogeneous boundary conditions,

$$G(a,\xi) = 0, \quad \frac{\partial G}{\partial \xi}(b,\xi) = 0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned}\left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x,\xi)-G(x,\xi)y'(\xi)\right)\right]_{\xi=a}^{\xi=b} &= \left[p(b)\left(y(b)\frac{\partial G}{\partial \xi}(x,b)-G(x,b)y'(b)\right)\right] \\ &\quad -\left[p(a)\left(y(a)\frac{\partial G}{\partial \xi}(x,a)-G(x,a)y'(a)\right)\right] \\ &= -\beta p(b)G(x,b)-\alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \quad (7.18)\end{aligned}$$

Therefore, we have the solution

$$y(x)=\int_a^b G(x,\xi)f(\xi)\,d\xi+\beta p(b)G(x,b)+\alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \quad (7.19)$$

This solution satisfies the non-homogeneous boundary conditions. Let's see how it works.

Example: Modify Example 8.4 to solve the boundary value problem $y'' = x^2$, $y(0) = 1$, $y(1) = 2$ using the boundary value Green's function that we found:

$$G(x,\xi)=\begin{cases}-\xi(1-x),\ 0\le\xi\le x\\ -x(1-\xi),\ x\le\xi\le 1\end{cases}. \quad (7.20)$$

We insert the Green's function into the solution and use the given conditions to obtain

$$\begin{aligned}y(x) &= \int_0^1 G(x,\xi)\xi^2\,d\xi-\left[y(\xi)\frac{\partial G}{\partial \xi}(x,\xi)-G(x,\xi)y'(\xi)\right]_{\xi=0}^{\xi=1} \\ &= \int_0^x (x-1)\xi^3\,d\xi+\int_x^1 x(\xi-1)\xi^2\,d\xi+y(0)\frac{\partial G}{\partial \xi}(x,0)-y(1)\frac{\partial G}{\partial \xi}(x,1) \\ &= \frac{(x-1)x^4}{4}+\frac{x(1-x^4)}{4}-\frac{x(1-x^3)}{3}+(x-1)-2x \\ &= \frac{x^4}{12}+\frac{35}{12}x-1. \quad (7.21)\end{aligned}$$

Of course, this problem can be solved more directly by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (7.12), can lead to the solution of boundary value problems. The Dirac delta function also aids in our interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the non homogeneous function is δ(x − ξ). Note that if we multiply the delta function by f(ξ) and integrate we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x).$$

We can view the delta function as a unit impulse at x = ξ which can be used to build f(x) as a sum of impulses of different strengths, f(ξ). Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{7.22}$$

Now, integrate both sides from ξ − ϙ to ξ + ϙ and take the limit as ϙ → 0. Then,

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi)\right] dx = \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx = 1. \tag{7.23}$$

Since the q(x) term is continuous, the limit of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon \to 0} \left[p(x) \frac{\partial G(x, \xi)}{\partial x} \right]_{\xi - \epsilon}^{\xi + \epsilon} = 1. \qquad (7.24)$$

This is the jump condition that we have been using!

Check your Progress-1

1. State any two properties of Green Function

___

___

___

2. Explain Direct Delta Function

___

___

___

3. Discuss Green's Function Differential Equation

___

___

___

7.5 LET US SUM UP

We have seen different properties of Green Function and its application. The Dirac delta function, as any distribution, only makes sense under an integral. We also explored Green's Function Differential Equation.

7.6 KEYWORDS

Differential Equation : an equation involving derivatives of a function or functions

Homogeneous Equation: A polynomial is **homogeneous** if all its terms have the same degree.

Notes

Nonhomogeneous differential **equations** : are the same as homogeneous differential **equations**, except they can have terms involving only x (and constants) on the right side, as in this **equation**

7.7 QUESTIONS FOR REVIEW

1. Evaluate

$$\int_0^{2\pi} \cos x\, \delta(x^2 - \pi^2)\, dx.$$

2. Evaluate $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx.$

7.8 SUGGESTED READINGS AND REFERENCES

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11. R. Churchil& J. Brown Fourier Series and Boundary Value Problems, McGraw-Hill, 1978
12. D. Powers, Boundary Value Problems Academic Press, 1979.

7.9 ANSWERS TO CHECK YOUR PROGRESS

1. Provide statement of properties – 7.2

2. Provide explanation– 7.3

3. Provide explanation– 7.4